CHAPTER 7 · FUNCTIONS

7.1 Introduction to Functions

What a function is · equality · examples · well-defined rules

Why functions?

Much of mathematics - and computing - is the study of rules that turn an input into a single output.

Arithmetic rules	$\mathrm{div}(n, d)$, $n \bmod d$, $\lvert x \rvert$, $\lfloor x \rfloor$
Relationships	fatherOf(x), capitalOf(country)
Computation	truthTable(p), hash(key), length(string)

input → **rule** → one output

In this course we focus on discrete functions - inputs and outputs drawn from discrete sets such as integers, strings, or finite sets.

7.1 Introduction to Functions

1. **What is a function**
2. Equality of functions
3. Examples of functions
4. Checking well-defined functions

Function, image, range, and inverse image

A function ***f* : *X* → *Y*** is a relation from the **domain** X to the **co-domain** Y satisfying two rules:

1. every element of X is related to some element of Y, and
2. no element of X is related to more than one element of Y.

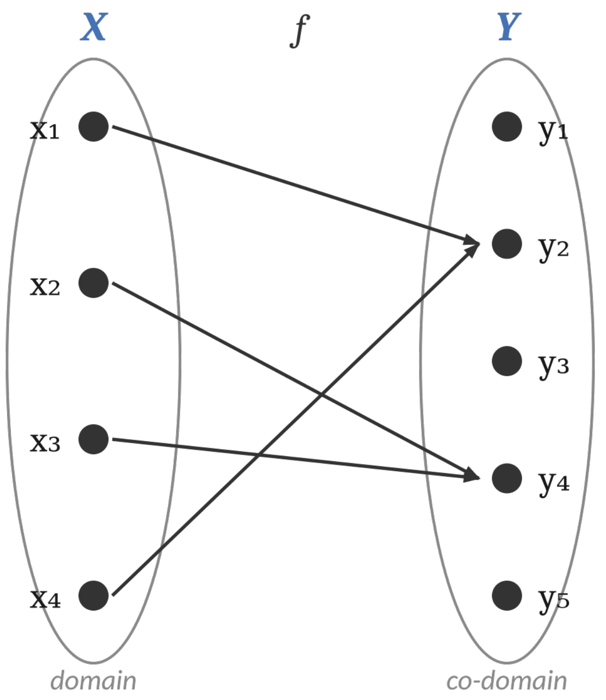
So each x in X has a **unique** image $f(x)$ in Y.

image / value of f at x: $f(x)$ - the unique output f sends x to.

range of f: $\{ y \in Y \mid y = f(x) \text{ for some } x \text{ in } X \}$ - outputs actually hit.

preimage of y: any x with f(x) = y.

inverse image of y: $\{ x \in X \mid f(x) = y \}$ - all such x.



range ⊆ co-domain: y_1, y_3, y_5 receive no arrow, yet f is still a function.

Notation and arrow diagrams

When X and Y are finite we can draw f as an arrow diagram: list X on the left, Y on the right, and draw one arrow out of each element of X.

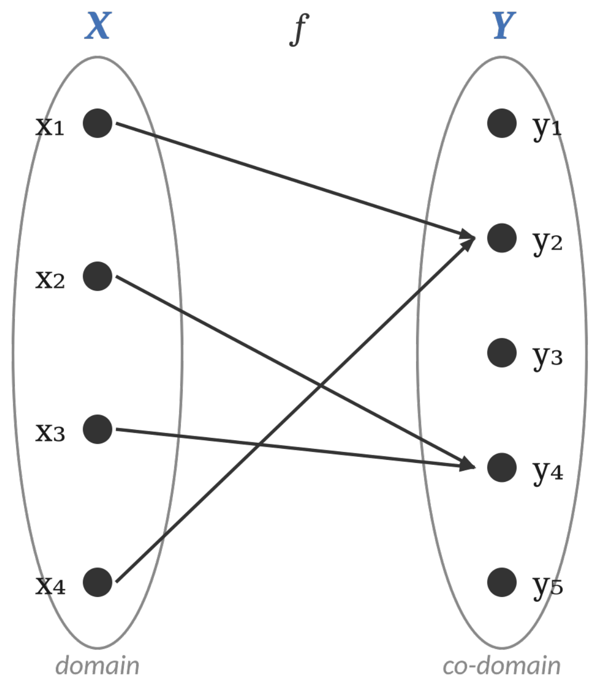
$f : X \to Y$ *"f is a function from X to Y"*

$f(x) = y$ *y is the image of x; f sends x to y*

$x \mapsto y$ *the "maps to" arrow (between elements)*

Two rules, in diagram language:

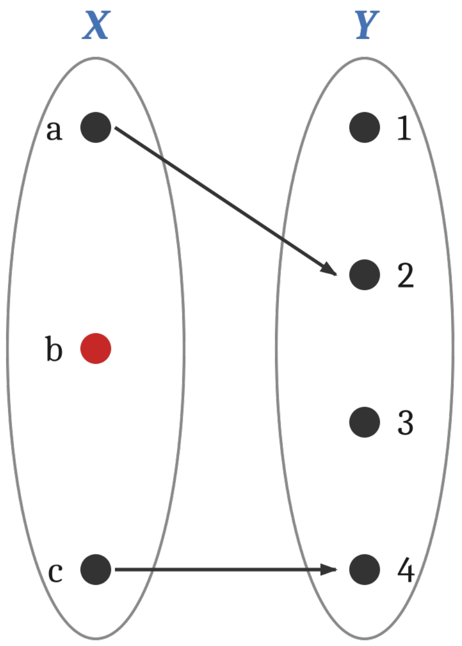
- every left element has an arrow leaving it (rule 1)
- no left element has two arrows leaving it (rule 2)



A right-hand element may receive zero, one, or many arrows - only the left side is constrained.

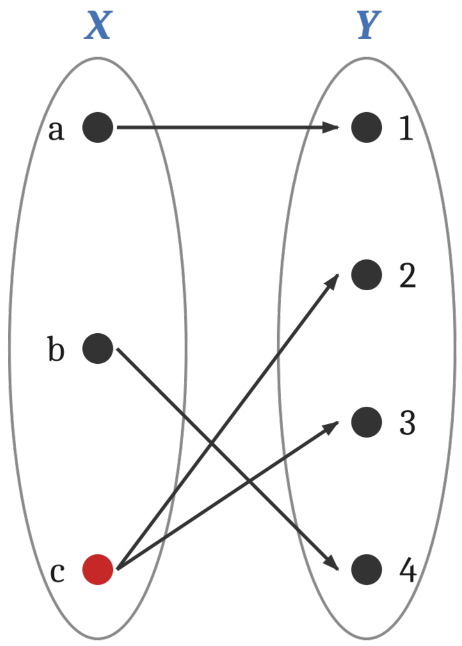
Example 7.1.1 - which arrow diagrams are functions?

X = {a, b, c}, Y = {1, 2, 3, 4}. Only one of these is a function.



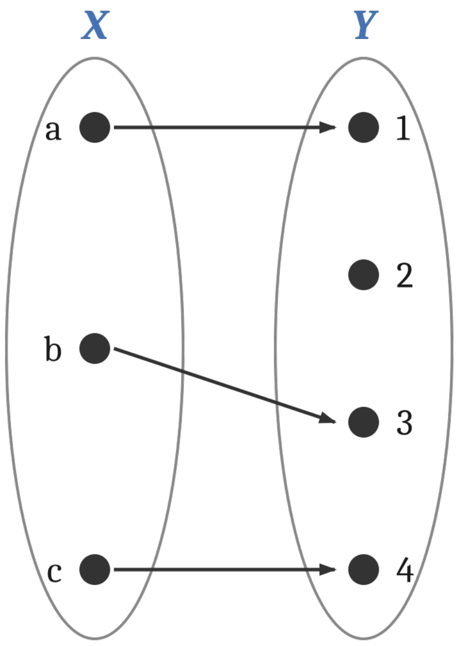
(a) **not a function**

b has no arrow (rule 1 fails)



(b) **not a function**

c has two arrows (rule 2 fails)



(c) **a function**

each input → exactly one output

Example 7.1.2 - reading a function off its diagram

Let X = {a, b, c} and Y = {1, 2, 3, 4}, with f given by the arrow diagram.

domain = {a, b, c}   co-domain = {1, 2, 3, 4}

f(a) = 2,   f(b) = 4,   f(c) = 2

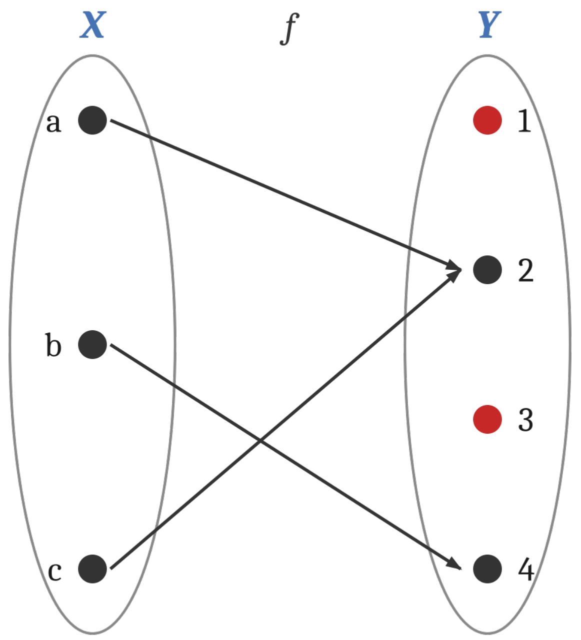
range = {2, 4}

inverse image of 2 = {a, c}

inverse image of 4 = {b}

inverse image of 1 = ∅   *(nothing maps to 1)*

f as ordered pairs = {(a, 2), (b, 4), (c, 2)}



Two arrows reach 2 (from a and c) - allowed. ***Nothing reaches 1 or 3*** *- also allowed.*

7.1 Introduction to Functions

1. What is a function
2. **Equality of functions**
3. Examples of functions
4. Checking well-defined functions

Equality of functions

Two functions are equal exactly when they have the same domain, the same co-domain, and agree on every input.

Theorem 7.1.1 (a test for function equality)

If $F: X \to Y$ and $G: X \to Y$ are functions, then $\mathbf{F = G \;\Leftrightarrow\; F(x) = G(x) \text{ for all } x \in X.}$

Why it holds (idea):

A function is its set of ordered pairs. Saying $(x, y) \in F$ is the same as saying $y = F(x)$. So F and G contain exactly the same pairs precisely when F(x) and G(x) match at every x - i.e. F = G as sets of pairs.

Consequence: the same rule written two different ways still gives *one* function - we just check it agrees at every input.

Example 7.1.3(a) - two rules, one function

On $J_3 = \{0, 1, 2\}$ define $f(x) = (x^2 + x + 1) \bmod 3$ and $g(x) = (x + 2)^2 \bmod 3$. **Does f = g?**

x	$x^2 + x + 1$	f(x)	$(x + 2)^2$	g(x)
0	1	1	4	1
1	3	0	9	0
2	7	1	16	1

The **f** and **g** columns are identical: **1, 0, 1**. By Theorem 7.1.1, **f = g** - even though the two rules look nothing alike.

Example 7.1.3(b) - F + G = G + F

For $F, G : R \to R$ define F + G and G + F by $(F + G)(x) = F(x) + G(x)$ and $(G + F)(x) = G(x) + F(x)$. **Is F + G = G + F ?**

$$\mathbf{(F + G)(x)} = F(x) + G(x) \quad \textit{by definition of F + G}$$

$$= G(x) + F(x) \quad \textit{addition of reals is commutative}$$

$$= (G + F)(x) \quad \textit{by definition of G + F}$$

This holds for every $x \in R$, so by Theorem 7.1.1, **F + G = G + F.**

Note. The commutativity comes from the real numbers, not from the functions. Where the underlying operation is not commutative, the built functions need not be equal.

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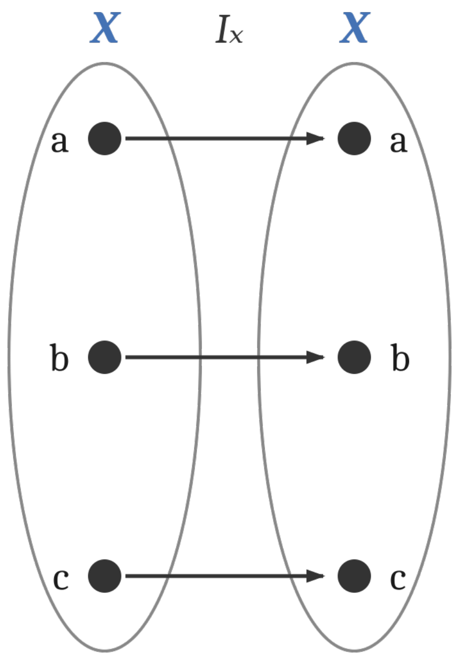
The identity function

For any set X, the **identity function** $I_X : X \to X$ is defined by

$$I_X(x) = x \quad \textit{for all } x \textit{ in } X.$$

It sends each element straight to itself — a machine that passes its input to the output unchanged. Its range equals its domain (all of X), and it is both one-to-one and onto.

Why we care: I_X is the “do nothing” function. It will return in 7.3 as the identity for composition: $f \circ I_X = f$, and $f^{-1} \circ f = I_X$.



A sequence is a function

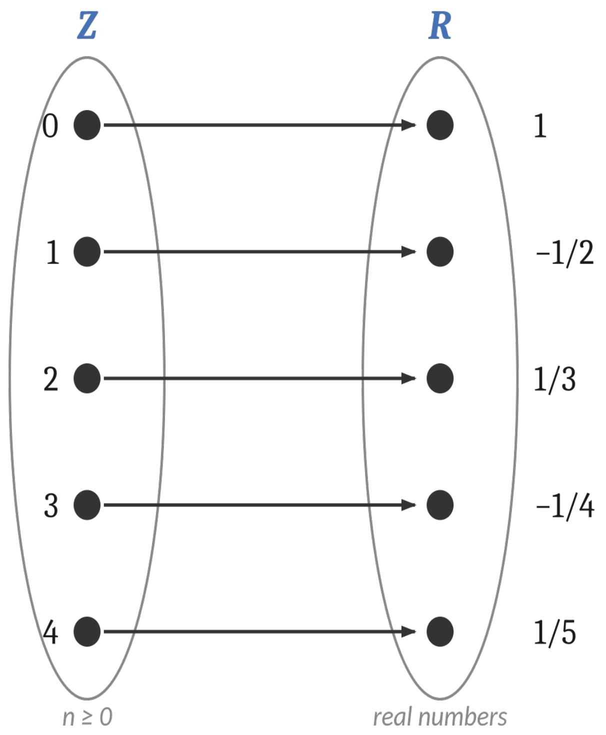
Formally…an infinite **sequence** is a function defined on the set of integers greater than or equal to a fixed integer.

The sequence

$$1,\ -1/2,\ 1/3,\ -1/4,\ 1/5,\ \ldots$$

is the function $f : Z_{\geq 0} \to R$ given by $f(n) = (-1)^n / (n + 1)$.

The same sequence, many functions. We may also index from 1: define $g : Z^+ \to R$ by $g(n) = (-1)^{n+1} / n$. Then $g(1) = 1$, $g(2) = -1/2$, … and $g(n + 1) = f(n)$: the two functions list the very same numbers, only the starting index differs.



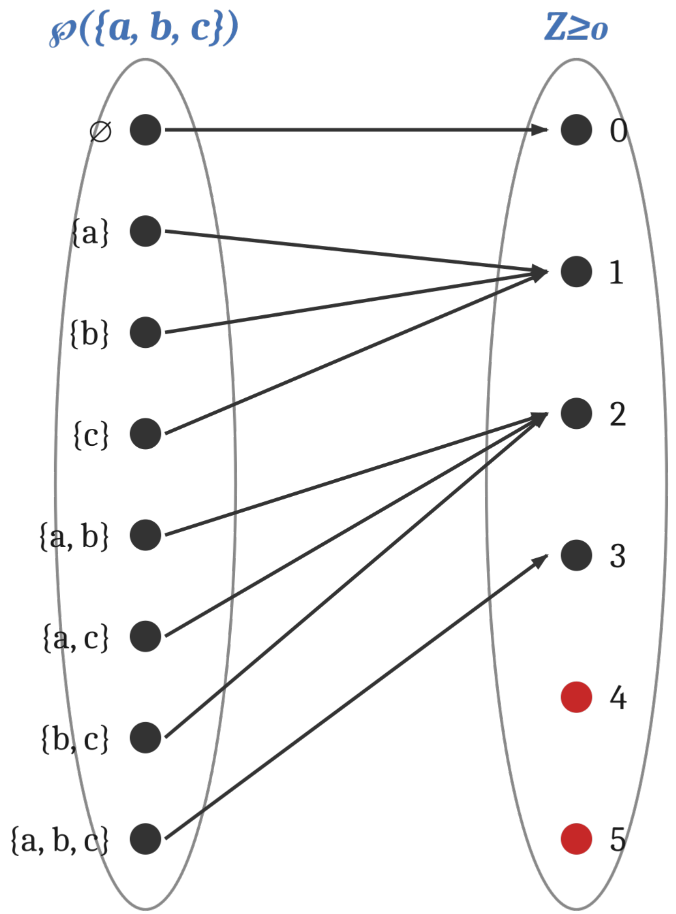
Example 7.1.6 — a function on a power set

Let P({a, b, c}) be the set of all subsets of {a, b, c}. Define

$$F : P(\{a, b, c\}) \to Z_{\geq 0}$$
$$F(X) = \text{the number of elements in } X.$$

Each subset maps to its size: ∅ → 0; the singletons → 1; the pairs → 2; {a, b, c} → 3.

Range vs co-domain again: the co-domain is all of $Z_{\geq 0}$, but the range is only {0, 1, 2, 3}. The values 4, 5, … are never reached — a 3-element set has no subset larger than 3.

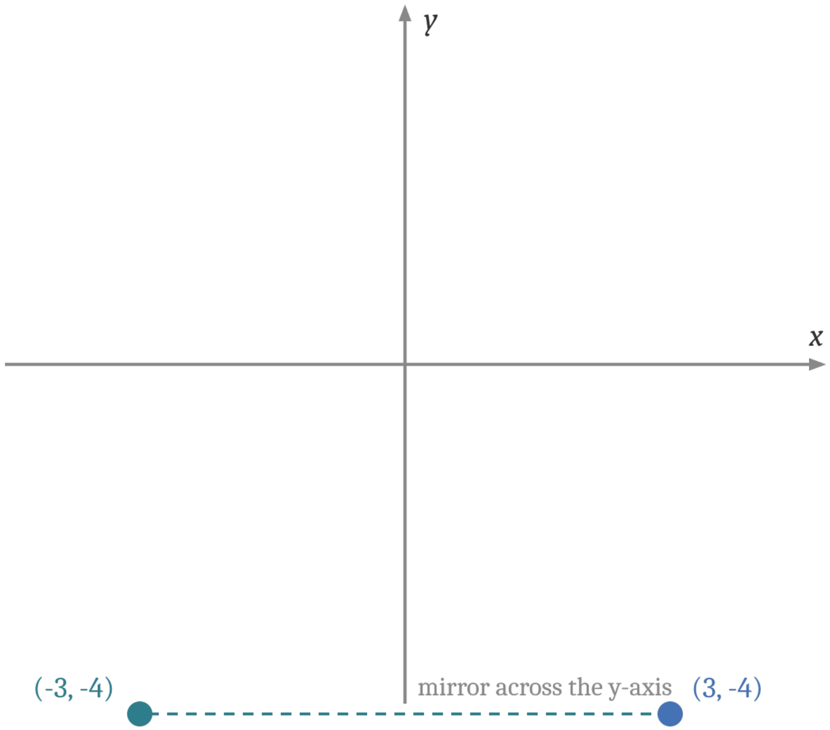


Example 7.1.7 — functions on a Cartesian product

Define M : R × R → R and R : R × R → R × R by, for all (a, b) in R × R,

$$M(a, b) = ab \qquad R(a, b) = (-a, b)$$

M multiplies the pair; **R** reflects the point across the vertical (y) axis.



$M(-1, -1) = 1$

$M(½, ½) = ¼$

$M(\sqrt{2}, \sqrt{2}) = 2$

$R(2, 5) = (-2, 5)$

$R(-2, 5) = (2, 5)$

$R(3, -4) = (-3, -4)$

Note. *We write M(a, b), dropping one set of parentheses around the input pair.*

A function on strings

Let S be the set of all strings of a's and b's (ε = the empty string). Define

$$g : S \to Z_{\geq 0} \qquad g(s) = \text{the number of a's in s.}$$

$g(\varepsilon) = 0$ *the empty string has no a's*

$g(bb) = 0$ *no a's*

$g(ababb) = 2$ *two a's*

$g(bbbaa) = 2$ *two a's*

Reading it as a function: the domain is the (infinite) set of strings, the co-domain is $Z_{\geq 0}$, and every string has exactly one a-count — so g is well defined. Strings are the basic objects of automata and formal-language theory.

Example 7.1.8 — the logarithmic function

For a base $b > 0$ with $b \neq 1$: $\log_b x$ = the exponent to which b must be raised to obtain x. $\log_b x = y \Leftrightarrow b^y = x$.

The logarithmic function with base b maps $R^+ \to R$. Evaluate:

$\mathbf{\log_3 9 = 2}$	*because $3^2 = 9$*
$\mathbf{\log_2 (1/2) = -1}$	*because $2^{-1} = 1/2$*
$\mathbf{\log_{10} 1 = 0}$	*because $10^0 = 1$*
$\mathbf{\log_2 (2^m) = m}$	*the exponent giving 2^m is m*
$\mathbf{2 \wedge (\log_2 m) = m}$	*$\log_2 m$ is the exponent giving m ($m > 0$)*

Main idea: "log" is just a name for "the exponent." Every logarithm question is secretly an exponent question. In 7.2 we will see $\log_b$ is the inverse of the exponential function b^x.

Example 7.1.11 — a Boolean function

A Boolean function takes tuples of 0's and 1's to {0, 1}. The three-place function

$$f : \{0, 1\}^3 \to \{0, 1\}$$

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \text{ mod } 2$$

This is the **parity** function: the output is 1 exactly when an odd number of inputs are 1.

Domain. $\{0, 1\}^3$ has $2^3 = 8$ triples — the eight rows of the table.

Why it matters. every logic circuit / truth table is a Boolean function. This connects 7.1 back to the digital-logic work in Chapter 2.

x_1	x_2	x_3	$f = (x_1+x_2+x_3)$ mod 2
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

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When a rule fails to define a function

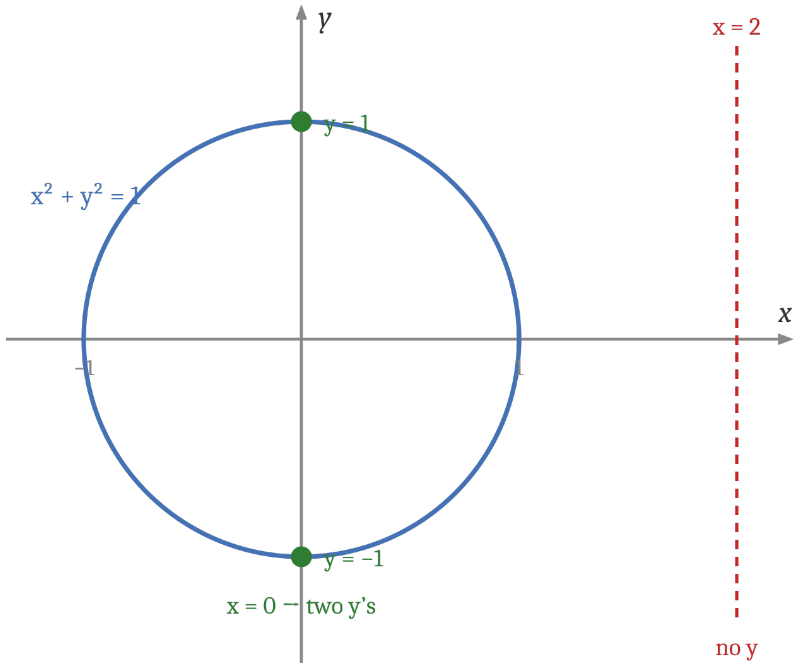
Try to define $f : R \to R$ by: f(x) is the real number y with $x^2 + y^2 = 1$.

This rule is not well defined — it breaks both function rules:

x = 2: no real y solves $4 + y^2 = 1$. *(rule 1 fails: an input with no image)*

x = 0: both y = 1 and y = −1 solve $0 + y^2 = 1$. *(rule 2 fails: two images)*

Main idea: a rule is **well defined** only if it gives every input exactly one unambiguous output. A rule that does not is simply not a function.



The vertical-line view: at x = 0 the line meets the circle twice; at x = 2 it misses entirely.

Example 7.1.12 — the same input, two names

Q is the set of rational numbers. Suppose we try to define

$$f : Q \to Z \qquad f(m\,/\,n) = m$$
(for all integers m, n with n ≠ 0)

Is f well defined? No — and here is the contradiction:

The same rational number has many names: $\mathbf{1/2 \;=\; 3/6.}$

But the formula reads off the numerator, so it gives different answers for the one number:

$$f(1/2) = 1 \quad \text{but} \quad f(3/6) = 3 \quad \Rightarrow \quad f(1/2) \neq f(3/6).$$

Contradiction: 1/2 and 3/6 are the *same* input, so a function must give them the same output. The rule does not — so f is not a function. (“well-defined function” is really redundant: being well defined is what makes it a function.)

Is it a function? — everyday relations

Take the relation from People to People. For each, ask the two rules: is everyone related to someone, and to only one?

Relation y = ___(x)	Function?	Why
FatherOf(x)	Yes	every person has exactly one (biological) father
MotherOf(x)	Yes	every person has exactly one (biological) mother
SonOf(x)	No	a person may have no son, or several — both rules can fail
BrotherOf(x)	No	a person may have zero or many brothers
WifeOf(x)	No	an unmarried person has none; the relation is not total

The test is always the same: total (everyone has an output) and single-valued (only one). Fail either, and it is a relation but not a function.

7.1 in one screen

A function $f : X \to Y$	is a relation in which every x in X has exactly one image f(x) in Y (total + single-valued).
Image, range, co-domain	image = the output f(x); range = all images; co-domain = the declared target. Range ⊆ co-domain, and the inclusion is often strict.
Preimage / inverse image	the inverse image of y is every x with f(x) = y; it may be empty, one, or many.
Equality (Thm 7.1.1)	same domain, same co-domain, and f(x) = g(x) for every x — then f = g.
Well defined	the rule must give one unambiguous output per input; if not, it is not a function.
Functions are everywhere	numbers, sequences, sets, pairs, strings, logarithms, circuits — all are functions.

Recommended exercises

Epp, 4th edition — Section 7.1 (p. 393)

2, 6, 7, 9, 18, 19, 20, 41, 42, 43

What they drill: reading domain / co-domain / range / inverse images off a diagram (2–9); equality of functions (18–20); and checking whether a rule is well defined (41–43).

Next: 7.2 — one-to-one and onto functions, bijections, inverses, and composition.