Chapter 6 - Set Theory

Part 1: Set Basics and Properties of Sets

Sections 6.1 - 6.2

COMP 233 Discrete Mathematics | Birzeit University

Where we are: Chapter 6 at a glance

6.1	**Set Basics** Sets, subsets, operations, partitions, power sets, products
6.2	**Properties of Sets** Set identities and the element method of proof
6.3	**Algebraic Proofs** Disproof by counterexample; deriving identities algebraically
6.4	**Boolean Algebra** The shared structure behind logic and sets

This deck covers 6.1 and 6.2.

6.1 Set Basics - outline

- 1. Basic concepts and notation
- 2. Subsets, proper subsets, set equality
- 3. Operations on sets
- 4. The empty set
- 5. Partitions of sets
- 6. Power sets and Cartesian products

What is a set?

A set is a collection of distinct objects. Each object in the set is called an element (or member).

Notation

$a \in S$ a is an element of S

$a \notin S$ a is not an element of S

Roster: $S = \{1, 2, 3\}$

Builder: $A = \{x \in S \mid P(x)\}$

"the set of all x in S such that P(x)"

Examples

$\{1, 2, 3, 4, 5\}$

$\{x \in Z \mid -2 < x < 5\} = \{-1, 0, 1, 2, 3, 4\}$

The set of all students born in Palestine

Three facts about set notation

1. **Order does not matter**

   $\{\text{Ali, Adam, Sara}\} = \{\text{Adam, Sara, Ali}\}$

2. **Repetition does not matter**

   $\{\text{Ali, Adam, Adam, Sara}\} = \{\text{Ali, Adam, Sara}\}$

3. **A set can be an element of another set**

   $\{1, \{1\}\}$ has two elements: 1 and $\{1\}$

Watch out: $\{\text{Ali}\} \neq \text{Ali}$ - a one-element set is not the same as its element.

Defining a set by a property

Form

$A = \{x \in S \mid P(x)\}$ *the set of all elements x of S that make the property P(x) true*

Worked example

Describe each set by listing its elements.

$E = \{x \in Z \mid x = 2k \text{ for some } k \in Z\}$ the even integers: $\{\ldots, -2, 0, 2, 4, \ldots\}$

$D = \{x \in R \mid 3 < x < 2\}$ no real number is both > 3 and < 2, so $D = \varnothing$

Subsets

Definition

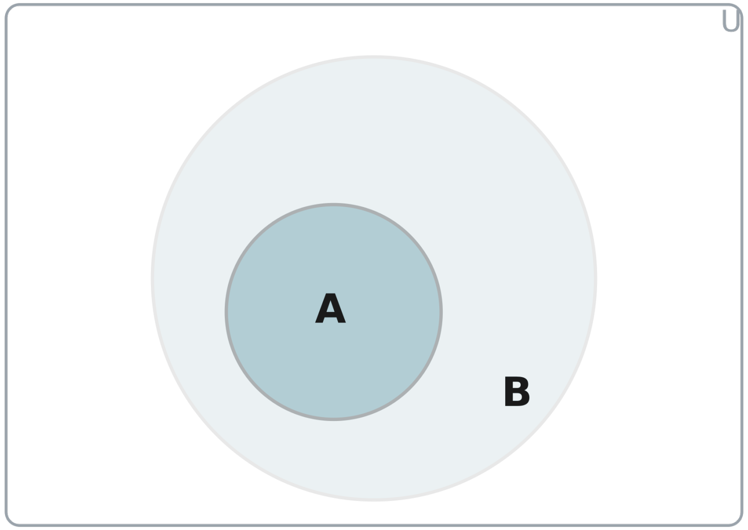
$A \subseteq B$ means every element of A is in B

$A \subseteq B \iff \forall x, \text{ if } x \in A \text{ then } x \in B$

Negation

$A \nsubseteq B$ means A is not a subset of B

$A \nsubseteq B \iff \exists x \text{ such that } x \in A \text{ and } x \notin B$



$A \subseteq B$

Do not confuse $\in$ with $\subseteq$

$\in$ relates an **element** to a set. $\subseteq$ relates a **set** to a set.

$2 \in \{1, 2, 3\}$	TRUE	$\{2\} \in \{1, 2, 3\}$	FALSE
$\{2\} \subseteq \{1, 2, 3\}$	TRUE	$2 \subseteq \{1, 2, 3\}$	FALSE
$\{2\} \in \{\{1\}, \{2\}\}$	TRUE	$\{2\} \subseteq \{\{1\}, \{2\}\}$	FALSE

Proper subsets and set equality

Proper subset

A is a proper subset of B when:

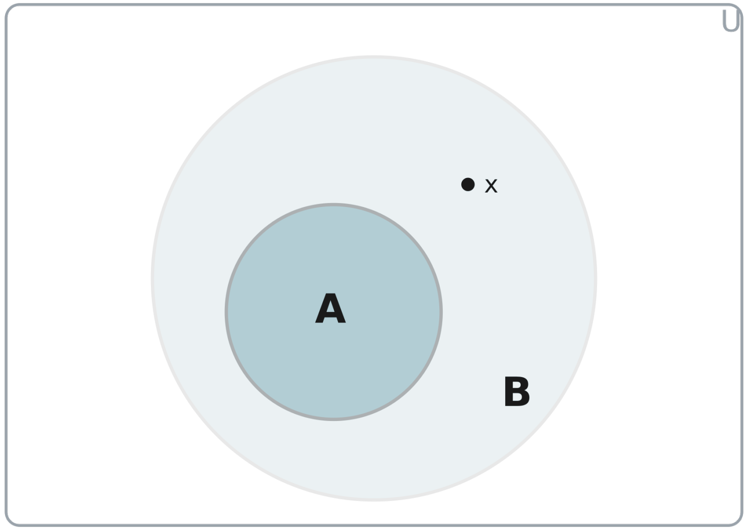
(1) $A \subseteq B$, and

(2) there is an element of B not in A

Set equality

Two sets are equal when each contains the other:

$$A = B \iff A \subseteq B \text{ and } B \subseteq A$$



$x \in B$ but $x \notin A$

The element argument: proving A ⊆ B

Method

To prove $A \subseteq B$:

1. Suppose x is a particular but arbitrarily chosen element of A.
2. Show that this x must also be an element of B.

Worked example (Epp 6.1.2)

$A = \{n \in Z \mid n = 6r + 12 \text{ for some } r \in Z\}$ $B = \{n \in Z \mid n = 3s \text{ for some } s \in Z\}$. Prove $A \subseteq B$.

Suppose $x \in A$. Then $x = 6r + 12$ for some integer r.

Then $x = 3(2r + 4)$. Let $s = 2r + 4$, an integer. So $x = 3s$, which means $x \in B$.

Since every element of A is in B, $A \subseteq B$.

Operations on sets (1): union and intersection

Union

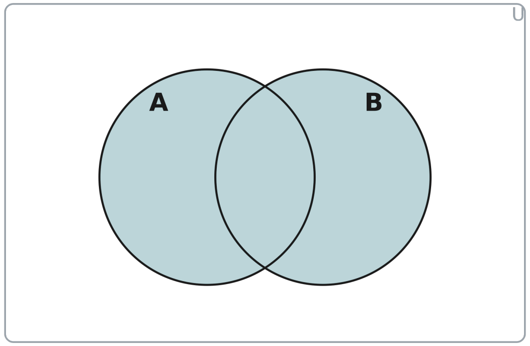
$A \cup B$ = all elements in A or B (or both)

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

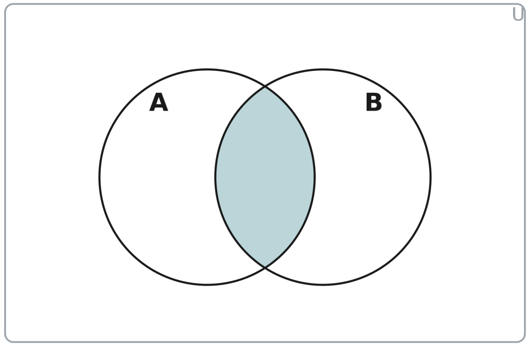
Intersection

$A \cap B$ = elements common to both

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



$A \cup B$



$A \cap B$

Operations on sets (2): difference and complement

Difference

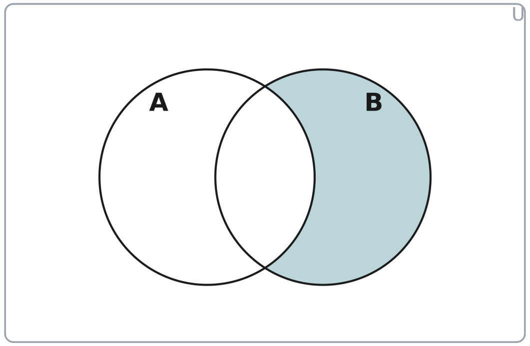
$B - A$ = in B but not in A

$B - A = \{x \mid x \in B \text{ and } x \notin A\}$

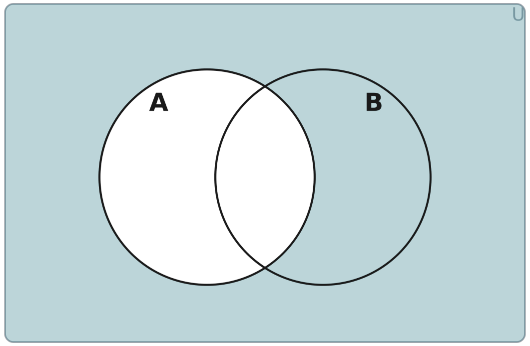
Complement

A^c = everything in U not in A

$A^c = \{x \in U \mid x \notin A\}$



$B - A$



A^c

Worked example: computing with operations

Given (Epp 6.1.5)

$U = \{a, b, c, d, e, f, g\}$ $A = \{a, c, e, g\}$ $B = \{d, e, f, g\}$

$A \cup B = \{a, c, d, e, f, g\}$

$A \cap B = \{e, g\}$

$B - A = \{d, f\}$

$A^c = \{b, d, f\}$

Intervals as sets of real numbers

Notation

$(a, b) = \{x \in R \mid a < x < b\}$ $[a, b] = \{x \in R \mid a \leq x \leq b\}$

Round bracket = endpoint excluded (open). Square bracket = endpoint included (closed).

Example (Epp 6.1.6): $A = (-1, 0]$ and $B = [0, 1)$

$A \cup B = (-1, 1)$ $A \cap B = \{0\}$ $B - A = (0, 1)$

Unions and intersections of many sets

Notation (a family of sets $A_1, A_2, \ldots, A_n$)

$A_1 \cup A_2 \cup \cdots \cup A_n = \{x \mid x \in A_i \text{ for } \textit{some } i\}$

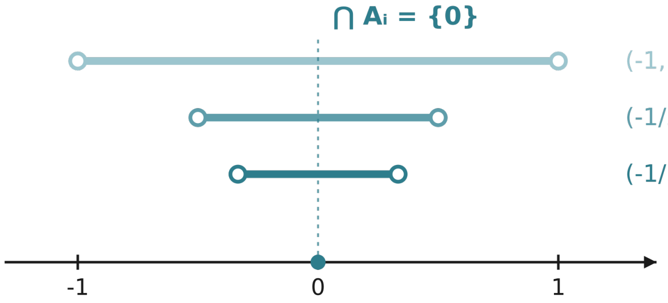
$A_1 \cap A_2 \cap \cdots \cap A_n = \{x \mid x \in A_i \text{ for } \textit{every } i\}$

Example (Epp 6.1.7)

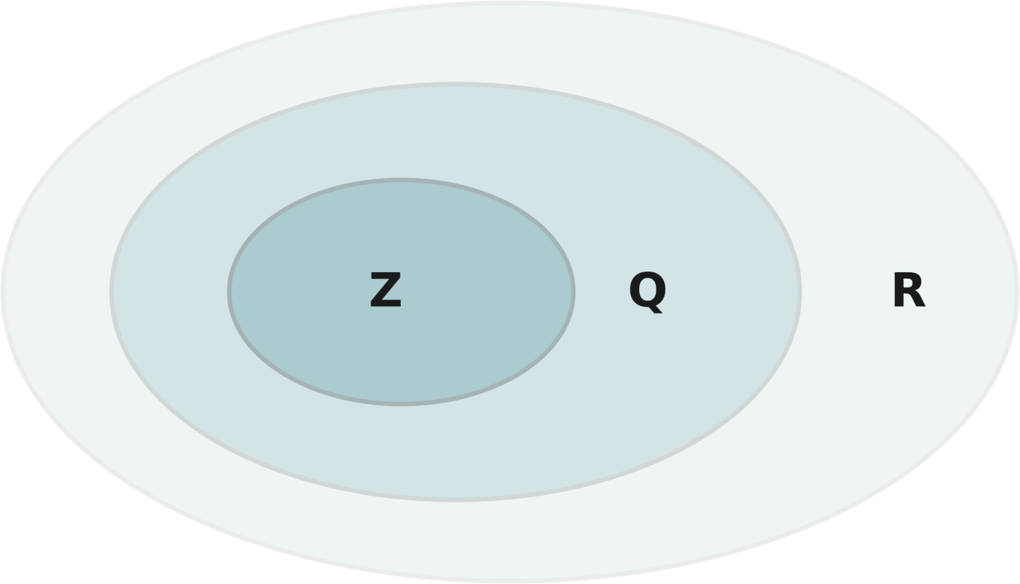
Let $A_i = (-1/i, 1/i)$ for $i = 1, 2, 3, \ldots$

$A_1 \cup A_2 \cup A_3 = (-1, 1)$

$A_1 \cap A_2 \cap A_3 = (-1/3, 1/3)$



A familiar chain of subsets: $Z \subseteq Q \subseteq R$



Why

Every integer is rational:

$n = n / 1$, so $Z \subseteq Q$

Every rational is real:

so $Q \subseteq R$

Proper, because:

$1/2 \in Q$ but $\notin Z$; $\sqrt{2} \in R$ but $\notin Q$

The empty set $\varnothing$

Definition

The empty set (or null set), written $\varnothing$, is the set with no elements.

Examples: $\{1, 3\} \cap \{2, 4\} = \varnothing$ $\{x \in R \mid x^2 = -1\} = \varnothing$

Two facts to remember

1. There is only ONE empty set (we prove this in 6.2).
2. $\varnothing$ is a subset of every set. And $\varnothing \neq \{\varnothing\}$: the right side has one element.

Partitions: cutting a set into disjoint pieces

Disjoint

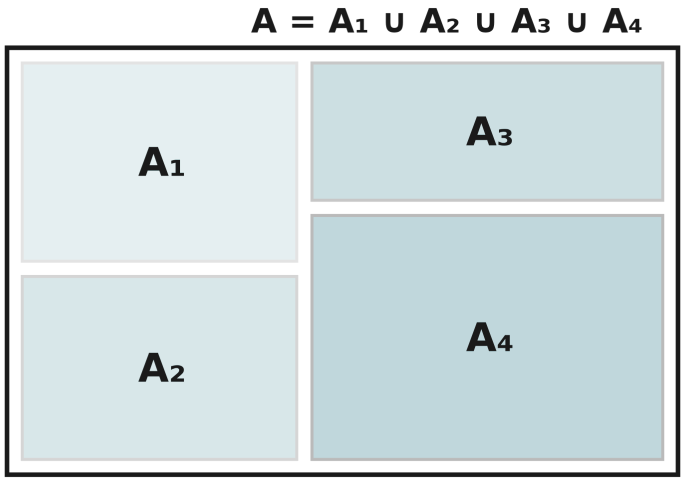
Sets are disjoint when they share no element:

A and B disjoint ⇔ $A \cap B = \varnothing$

Partition

$\{A_1, A_2, \ldots\}$ is a partition of A when:

1. their union is all of A, and
2. they are mutually disjoint (no two overlap).



Example: $Z = (3k) \cup (3k+1) \cup (3k+2)$

Power sets

Definition

The power set $P(A)$ is the set of ALL subsets of A.

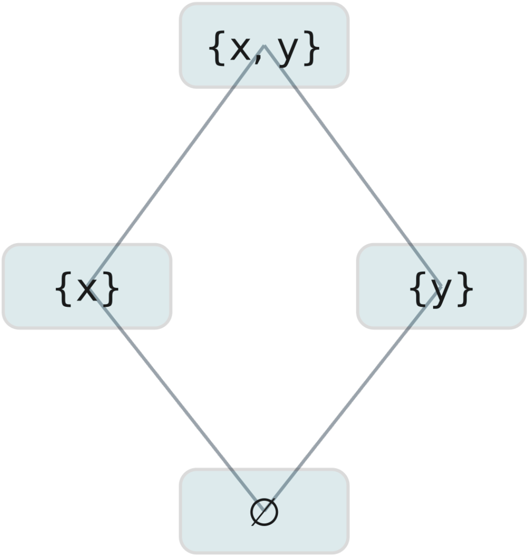
If A has n elements, P(A) has 2^n elements (proved in 6.3).

Example (Epp 6.1.12)

$P(\{x, y\}) =$

$\{ \varnothing, \{x\}, \{y\}, \{x, y\} \}$

4 subsets, and $2^2 = 4$.



Cartesian products

Definition

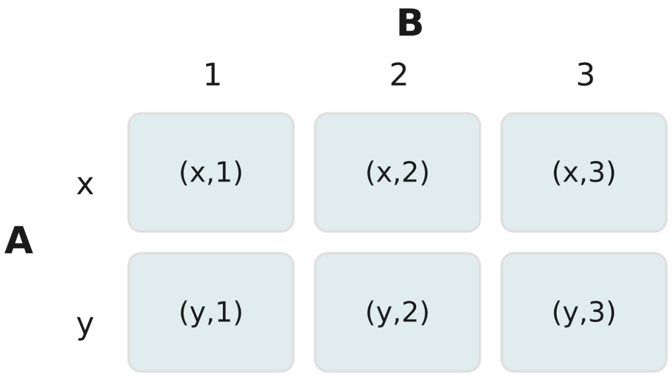
An ordered pair (a, b) keeps order: (a, b) = (c, d) only if a = c and b = d.

$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

Example (Epp 6.1.14)

$A = \{x, y\}, \quad B = \{1, 2, 3\}$

$A \times B$ has $2 \times 3 = 6$ pairs



A \ **B**	1	2	3
x	(x,1)	(x,2)	(x,3)
y	(y,1)	(y,2)	(y,3)

6.2 Properties of Sets - outline

- 1. Subset relations and the catalogue of set identities
- 2. Proving set identities by the element method
- 3. Worked proofs: distributive, De Morgan, and empty-set proofs

Goal: prove set facts rigorously, not just by drawing a Venn diagram.

Subset relations and procedural definitions

Theorem 6.2.1 (some subset relations)

$A \cap B \subseteq A \quad A \cap B \subseteq B$

$A \subseteq A \cup B \quad B \subseteq A \cup B$

Transitivity:

if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Procedural definitions (the proof engine)

$x \in X \cup Y \iff x \in X$ or $x \in Y$

$x \in X \cap Y \iff x \in X$ and $x \in Y$

$x \in X - Y \iff x \in X$ and $x \notin Y$

$x \in X^c \iff x \notin X$

$(x,y) \in X \times Y \iff x \in X$ and $y \in Y$

Theorem 6.2.2: the set identities

These hold for all subsets of a universal set U. They are our toolkit for the rest of the chapter.

Commutative	$A \cup B = B \cup A, \quad A \cap B = B \cap A$	**Idempotent**	$A \cup A = A, \quad A \cap A = A$
Associative	$(A \cup B) \cup C = A \cup (B \cup C)$	**Universal bound**	$A \cup U = U, \quad A \cap \varnothing = \varnothing$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	**De Morgan**	$(A \cup B)^c = A^c \cap B^c$
Identity	$A \cup \varnothing = A, \quad A \cap U = A$	**Absorption**	$A \cup (A \cap B) = A$
Complement	$A \cup A^c = U, \quad A \cap A^c = \varnothing$	**Comp. of U, ∅**	$U^c = \varnothing, \quad \varnothing^c = U$
Double comp.	$(A^c)^c = A$	**Set difference**	$A - B = A \cap B^c$

Proving two sets are equal

Basic method

To prove $X = Y$, prove BOTH containments:

(1) $X \subseteq Y$ (2) $Y \subseteq X$

Worked example (Epp 6.2.1): prove $A \cap B \subseteq A$

Suppose x is any element of $A \cap B$.

By the procedural definition of intersection, $x \in A$ and $x \in B$.

In particular, $x \in A$.

Since every element of $A \cap B$ is in A, $A \cap B \subseteq A$.

Worked proof: a distributive law

Prove: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Epp 6.2.2)

Proof of $\subseteq$ (one direction shown)

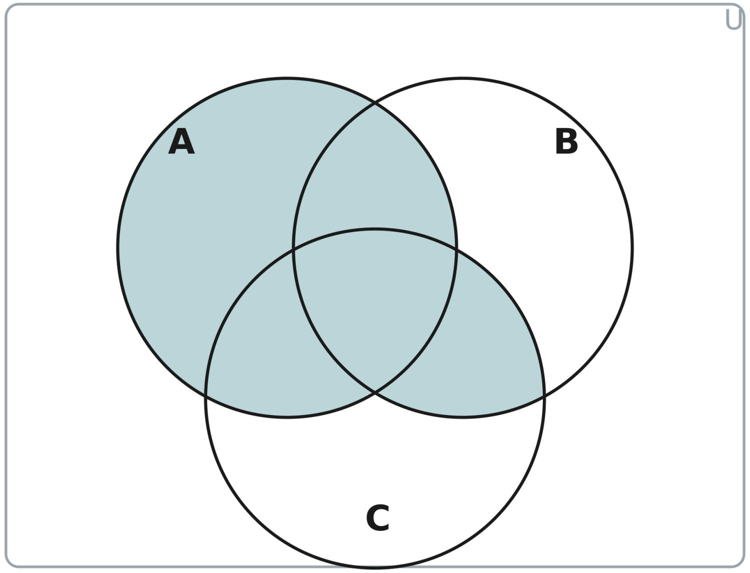
Suppose $x \in A \cup (B \cap C)$.

Then $x \in A$, or $x \in B \cap C$.

Case 1 ($x \in A$): then $x \in A \cup B$ and $x \in A \cup C$, so x is in their intersection.

Case 2 ($x \in B \cap C$): then $x \in B$ and $x \in C$, so $x \in A \cup B$ and $x \in A \cup C$.

Either way, $x \in (A \cup B) \cap (A \cup C)$.



$A \cup (B \cap C)$ shaded

Worked proof: a De Morgan law

Prove: $(A \cup B)^c = A^c \cap B^c$ (Epp 6.2.3)

Proof of ⊆ (one direction shown)

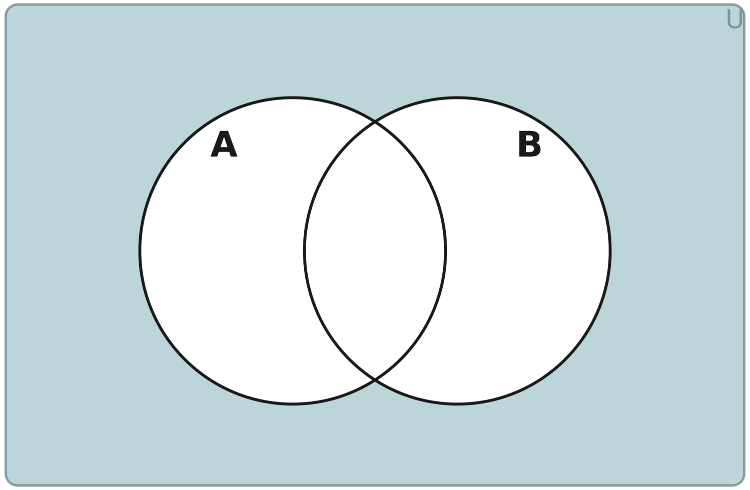
Suppose $x \in (A \cup B)^c$.

Then $x \notin A \cup B$, i.e. NOT($x \in A$ or $x \in B$).

By De Morgan's law of LOGIC:

$x \notin A$ and $x \notin B$.

So $x \in A^c$ and $x \in B^c$, hence $x \in A^c \cap B^c$.



$(A \cup B)^c$ shaded

Proving that a set is empty

Method (proof by contradiction)

To prove $X = \varnothing$: suppose X has an element x, and derive a contradiction.

This works because there is only one empty set, so “no elements” forces $X = \varnothing$.

Worked example (Epp 6.2.4): prove $A \cap \varnothing = \varnothing$

Suppose, for contradiction, that $A \cap \varnothing$ has an element x.

Then $x \in A$ and $x \in \varnothing$, by definition of intersection.

But $x \in \varnothing$ is impossible: the empty set has no elements. Contradiction.

So $A \cap \varnothing$ has no element, hence $A \cap \varnothing = \varnothing$.

Recommended exercises (Epp, 4th ed.)

6.1 Subsets: prove or disprove	3, 4, 5, 6, 7
6.1 Set operations & Venn diagrams	10, 11, 12, 13, 14, 17
6.1 Partitions	27, 28, 29, 30
6.1 Power sets & Cartesian products	31, 32, 33, 34, 35
6.2 Element-argument proofs	5, 7, 8, 9, 11, 13, 14, 15
6.2 Find the mistake (build rigor)	20, 21, 22
6.2 Proving a set is empty	27, 28, 29, 30, 31

Summary

- A set is a collection of distinct elements; describe it by roster or by a property.
- $A \subseteq B$ means "for all x, if $x \in A$ then $x \in B$"; $A = B$ means $A \subseteq B$ and $B \subseteq A$.
- Operations: union (or), intersection (and), difference, complement (within a universe U).
- Partitions split a set into mutually disjoint pieces; P(A) collects all subsets (2^n of them).
- Element method: to prove a set fact, take an arbitrary element and unfold the definitions.
- Set identities mirror the logical equivalences - the bridge to Boolean algebra in 6.4.