# Chapter 5: Sequences & Mathematical Induction

## Part 2: 5.2 & 5.3 Mathematical Induction

COMP 233 Discrete Mathematics | Birzeit University

*Content follows Epp, Discrete Mathematics with Applications (4th ed.).*

# 5.2 & 5.3 Induction - in this lecture

1 **What induction is, and why it works**

2 The method, and a first full proof

3 Sums: Gauss, arithmetic and geometric formulas

4 Divisibility and inequalities

5 Properties of recursive sequences

6 Induction vs. deduction, and more practice

# A puzzle that needs a new kind of proof

**The question**

Using only 3-cent and 5-cent coins, which total amounts n can you pay exactly?

$$8 = 3 + 5, \quad 9 = 3 + 3 + 3, \quad 10 = 5 + 5, \quad 11 = 3 + 3 + 5$$

**The pattern, and the problem**

Every amount from 8 upward seems to work - but "8, 9, 10, 11, ... and onward forever" is infinitely many cases.

We cannot check them one by one. We need a method that proves a statement for all integers $n \geq 8$ at once. That method is mathematical induction.

# The principle of mathematical induction

**To prove P(n) is true for all integers n ≥ a**

**Basis step**

$$(1)\ \text{Basis}: \quad P(a) \text{ is true}$$

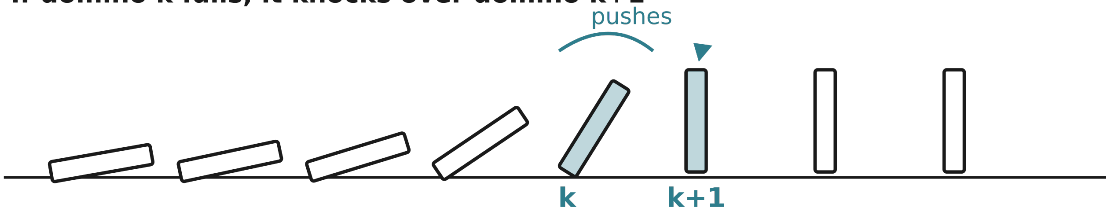
**Inductive step**

$$(2)\ \text{Inductive step}: \quad \forall k \geq a, \quad P(k) \Rightarrow P(k+1)$$

**Conclusion**

$$\text{Then } P(n) \text{ is true for every integer } n \geq a$$

**If domino k falls, it knocks over domino k+1**




*Basis topples the first domino; the inductive step makes each falling domino topple the next.*

# One method, four kinds of statement

### Sums

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

*rewrite the sum as a closed form*

### Divisibility

$$3 \mid (2^{2n} - 1)$$

*show the expression = d k for some integer k*

### Inequalities

$$2n + 1 < 2^n$$

*start the basis where it first holds*

### Recursive sequences

$$a_n = 2 \cdot 5^{\,n-1}$$

*use the recurrence as the link*

**The same basis + inductive step proves all four - only the algebra in the step changes.**

# 5.2 & 5.3 Induction - in this lecture

# Anatomy of an induction proof

### The three moving parts

1. **Basis step:** prove the statement directly at the starting value (often $n = 1$ or $n = 0$).
2. **Inductive hypothesis:** **ASSUME** the statement holds for one integer $k \geq a$.
3. **Inductive step:** USING that assumption, prove the statement for $k + 1$.

### A common worry, settled

"Isn't assuming $P(k)$ the same as assuming what we want?" No. We assume it for a single $k$ and show that forces $P(k+1)$. We never assume $P(n)$ for all $n$. The basis plus that single link is what makes the whole chain fall.

# The induction recipe - a reusable template

| | |
|---|---|
| **1. State** | Write the statement P(n) you want to prove, and the starting value a. |
| **2. Basis** | Check P(a) directly at the smallest n - usually one line of arithmetic. |
| **3. Hypothesis** | Assume P(k) holds for one arbitrary integer $k \geq a$. Write it out explicitly. |
| **4. Step** | Starting from P(k), derive P(k+1) by algebra. This is the real work. |
| **5. Conclude** | By induction, P(n) is true for every integer $n \geq a$. |

# Worked proof: the coin problem

**Claim** (Epp, Proposition 5.2.1): every amount $n \geq 8$ cents can be paid using 3¢ and 5¢ coins.

**Proof by induction on the amount n**

$P(n)$: "amount n can be paid with 3¢ and 5¢ coins." *n is the amount itself – there is no sequence, so nothing is being indexed.*

**Basis step.** $P(8)$: $8 = 3 + 5$ (one 3¢ coin and one 5¢ coin).

**Inductive step.** Take any integer $k \geq 8$ and assume $P(k)$:

$k = 3a + 5b$ for some integers $a, b \geq 0$ (a = number of 3¢ coins, b = number of 5¢ coins).

**Goal:** write $k + 1$ again in the form $3(\cdot) + 5(\cdot)$.

*There is no 1¢ coin, so the only way to raise the total by exactly 1 is to trade coins whose values differ by 1.*

**Case 1.** the payment has a 5¢ coin ($b \geq 1$)

Trade one 5¢ coin for two 3¢ coins.

Net change: $+6 - 5 = +1$

$$k + 1 = 3(a + 2) + 5(b - 1)$$

**Case 2.** the payment is only 3¢ coins ($b = 0$)

Then $3a \geq 8$, so $a \geq 3$: at least three 3¢ coins. Trade three 3¢ for two 5¢.

Net change: $+10 - 9 = +1$

$$k + 1 = 3(a - 3) + 5(b + 2)$$

Either way k + 1 is paid with 3¢ and 5¢ coins, so $P(k + 1)$ is true.

# 5.2 & 5.3 Induction - in this lecture

# Gauss and the power of a closed form

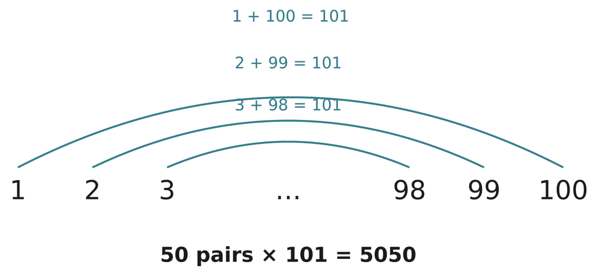
**The story**

Asked to add 1 + 2 + … + 100, the young Gauss paired the ends: 1 + 100, 2 + 99, 3 + 98, … Fifty pairs, each summing to 101.

$$1 + 2 + 3 + \cdots + 100 \quad = \quad 5050$$

↑ *the sum* ↑ *the total*

*A closed form gives the total directly - no loop, no adding term by term.*



# From a sum to a closed form

### The sum (expanded form)

$$1 + 2 + 3 + \cdots + n$$

Add every term, one by one. To reach the answer you do n additions - the work grows with n.

### The closed form

$$\frac{n(n+1)}{2}$$

One formula in n: multiply, add, divide. The same three operations for any n - no loop at all.

### The relationship

**Same number, two expressions.** The sum says ***what*** to compute; the closed form says ***how*** to get it directly. They are equal for every n - and induction is exactly the proof that they always agree.

# Closed forms: why we want them, and how to find one

## Why a closed form is worth having

| | |
|---|---|
| **Speed** | one calculation, constant work - the sum needs n steps, the formula needs one. |
| **Reach** | evaluate it for any n at once, even n = 1,000,000,000. |
| **Insight** | it exposes the growth rate at a glance: |

$$\frac{n(n+1)}{2} \approx \frac{n^2}{2}$$

*grows like $n^2$ - quadratic*

## How we find one

1. Spot a pattern or use a trick - Gauss pairing, or telescoping.
2. Conjecture the closed form, and check it on a few small n.
3. Prove it holds for all n by induction - turning the guess into a theorem.

# Theorem 5.2.2: sum of the first n integers

**For every integer n ≥ 1**

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

↑ *the sum, written out* ↑ *the closed form*

**The same thing, as a loop**

```
s = 0;   for (k = 1; k ≤ n; k++)   s = s + k;
```

The loop adds the integers one at a time in n steps. The closed form n(n+1)/2 returns the same answer in a single step - and induction proves the two always agree.

# Proof of Theorem 5.2.2, step by step

$$\frac{n(n+1)}{2}$$

**Basis (n = 1)** $\quad n=1:\quad 1=\frac{1\cdot 2}{2}$

**Assume P(k)** $\quad 1+2+\cdots+k=\frac{k(k+1)}{2}$

**Inductive step - start at the left side and reach the goal**

*must reach:* $\quad 1+2+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}$

*start (LHS at k+1)* $\quad 1+2+\cdots+k+(k+1)$

*use P(k)* $\quad =\frac{k(k+1)}{2}+(k+1)$

*common denominator* $\quad =\frac{k(k+1)+2(k+1)}{2}$

*factor out (k+1)* $\quad =\frac{(k+1)(k+2)}{2}$

# Using the formula to evaluate sums

**Sum of even numbers - factor the 2 out first**

$$2 + 4 + \cdots + 500 = 2 \cdot \frac{250 \cdot 251}{2} = 62750$$

$$\frac{n(n+1)}{2}$$

**Sum from 5 to 50 - whole sum minus the missing head**

$$5 + 6 + \cdots + 50 = \frac{50 \cdot 51}{2} - \frac{4 \cdot 5}{2} = 1265$$

# Theorem 5.2.3: the geometric sum

**For a fixed ratio $r \neq 1$ and integer $n \geq 0$**

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

↑ *the geometric sum* ↑ *the closed form*

**Why it matters**

A geometric sum adds powers of a fixed ratio r: $1 + r + r^2 + \ldots + r^n$ - compound interest, loop-doubling costs, binary place values. The closed form collapses the whole sum into one fraction.

**Example (ratio r = 2)**

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

# Proof of Theorem 5.2.3, step by step

**Basis (n = 0)** $$n=0:\quad r^0=1=\frac{r^1-1}{r-1}$$

**Assume P(k)** $$\sum_{i=0}^{k} r^i=\frac{r^{k+1}-1}{r-1}$$

**Inductive step - split off the last term, then combine over r – 1**

*must reach (we need to show):* $$\sum_{i=0}^{k+1} r^i=\frac{r^{k+2}-1}{r-1}$$

*split off r^(k+1)* $$\sum_{i=0}^{k+1} r^i=\sum_{i=0}^{k} r^i+r^{k+1}$$

*use P(k)* $$=\frac{r^{k+1}-1}{r-1}+r^{k+1}$$

*common denominator* $$=\frac{r^{k+1}-1+r^{k+1}(r-1)}{r-1}$$

*simplify numerator* $$=\frac{r^{k+2}-1}{r-1}$$

# 5.2 & 5.3 Induction - in this lecture

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2 The method, and a first full proof

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**4 Divisibility and inequalities**

5 Properties of recursive sequences

6 Induction vs. deduction, and more practice

# Divisibility proof, step by step (Prop 5.3.1)

**Claim:** $3 \mid (2^{2n} - 1) \quad (n \geq 0)$

*course notation:* $d \mid n \Leftrightarrow n = dk \ \ (k \in \mathbb{Z})$

**Basis (n = 0)** $n = 0: \quad 2^0 - 1 = 0 = 3 \cdot 0$

**Assume P(k)** $2^{2k} - 1 = 3r$

**Inductive step - peel out the previous block, then exhibit a multiple of 3**

*expand the power* $2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1$

*peel out the block* $= 4(2^{2k} - 1) + 3$

*use P(k): ... = 3r* $= 4(3r) + 3$

*factor out 3* $= 3(4r + 1)$

# Inequality proof, step by step (Prop 5.3.2)

$2n + 1 < 2^n$

**Basis (n = 3)** $n = 3: \quad 2(3) + 1 = 7 < 8 = 2^3$

**Assume P(k)** $2k + 1 < 2^k$

**Inductive step - rewrite, use P(k), then bound $2 \le 2^k$ ($k \ge 3$)**

| | |
|---|---|
| *must reach:* | $2(k + 1) + 1 < 2^{k+1}$ |
| *rewrite the Left HAND SIDE* | $2(k + 1) + 1 = (2k + 1) + 2$ |
| *use P(k)* | $< 2^k + 2$ |
| *bound: $2 \le 2^k$* | $\le 2^k + 2^k$ |
| *combine* | $= 2^{k+1}$ |

**Reading the chain**

Each symbol links its line to the one above it. Read end to end:

$2(k+1)+1 = (2k+1)+2 < 2^k+2 \le 2^k+2^k = 2^{k+1}$

**Rule:** a chain of =, ≤ and at least one < is overall < (strict). You don't need every link to be strict.

*One strict step (using P(k)) makes the whole chain strict, so:*

$2(k + 1) + 1 < 2^{k+1}$

# 5.2 & 5.3 Induction - in this lecture

# Recursive-sequence proof, step by step

**The recurrence, and the closed form to prove**

$a_1 = 2, \quad a_k = 5\, a_{k-1} \quad (k \ge 2)$

**Claim:** $a_n = 2 \cdot 5^{n-1}$

**Basis (n = 1)** $n = 1: \quad a_1 = 2 = 2 \cdot 5^0$

**Assume P(k)** $a_k = 2 \cdot 5^{k-1}$

**Inductive step - use the recurrence, then P(k)**

*by the recurrence* $a_{k+1} = 5\, a_k$

*use P(k)* $= 5\,(2 \cdot 5^{k-1})$

*regroup* $= 2 \cdot (5 \cdot 5^{k-1})$

*simplify* $\mathbf{= 2 \cdot 5^k}$

***Which is what we need to show at n = k+1:*** $2 \cdot 5^{(k+1)-1}$

**Reading the closed form**

$2 \cdot 5^{n-1}$ means $2 \times (5^{n-1})$

not $(2 \cdot 5)^{n-1} = 10^{n-1}$ *(wrong)*

**2** the start, $a_1$ - stays out front

**5** the ratio - ×5 each step

**n−1** how many ×5 steps so far

*Watch the 2 ride along:*

$a_1 = 2 \cdot 5^0 = 2$

$a_2 = 2 \cdot 5^1 = 10$

$a_3 = 2 \cdot 5^2 = 50$

$a_4 = 2 \cdot 5^3 = 250$

*Same 2 every line; only the power of 5 climbs.*

# 5.2 & 5.3 Induction - in this lecture

1. What induction is, and why it works
2. The method, and a first full proof
3. Sums: Gauss, arithmetic and geometric formulas
4. Divisibility and inequalities
5. Properties of recursive sequences
6. **Induction vs. deduction, and more practice**

# Inductive reasoning vs. deductive reasoning

### Inductive reasoning

Starts from **observations and examples**.

Moves from **specific TO —> general**.

Produces a **conjecture** - it suggests, it does not prove.

"We had a quiz each lecture in the past months, so we will have a quiz next lecture"

### Deductive reasoning

Starts from **definitions and theorems**.

Moves from **general —> specific**.

Produces a **proof** - the conclusion is guaranteed.

"If my highest mark this semester is 82%, then my average will not be more than 82%"

**Watch the name:** despite the word "induction", mathematical induction is a **deductive** method - it proves, with certainty, for all n.

# Common mistakes to avoid

| | |
|---|---|
| **Skipping the basis** | Without the first domino nothing falls - the step alone proves nothing. |
| **Wrong starting value** | Check where the claim first holds: $2n+1 < 2^n$ only from $n = 3$, not $n = 1$. |
| **Misreading the hypothesis** | You assume $P(k)$ for one $k$, not $P(n)$ for all $n$. That is not circular. |
| **Off-by-one in the step** | Be sure what the $(k+1)$-th term actually is before adding it on. |
| **Stopping early** | Reaching the closed form is not enough - state the conclusion for all $n$. |

# More statements you can now prove by induction

*Each follows the same recipe - basis, hypothesis, step. Sums and divisibility and inequalities, all in one toolkit:*

| | |
|---|---|
| **Sum formula** | $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ |
| **Sum of cubes** | $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ |
| **Divisibility** | $5 \mid (8^n - 3^n)$ |
| **Divisibility** | $3 \mid (n^3 + 2n)$ |
| **Inequality** | $n^2 < 2^n \quad (n \geq 5)$ |

# Recommended exercises (Epp, 4th ed.)

| | |
|---|---|
| 5.2 Model a problem like the coins / postage | **1, 2** |
| 5.2 Prove a sum formula directly | **6, 7, 8, 9** |
| 5.2 Prove the standard formulas ($\Sigma i^2$, $\Sigma i^3$, …) | **10, 11, 12, 13, 17** |
| 5.2 Evaluate a sum using the theorems | **20, 21, 22, 23, 25, 26** |
| 5.3 Divisibility by induction | **8, 9, 10, 11, 12, 14, 15** |
| 5.3 Inequalities by induction | **16, 17, 18, 19, 20, 23** |
| 5.3 Closed form of a recursive sequence | **24, 25, 26, 27** |

# Summary

- Mathematical induction proves a statement for all integers $n \geq a$ using two steps: a basis case, and an inductive step $P(k) \Rightarrow P(k+1)$.
- The inductive hypothesis is the assumption $P(k)$ for one $k$; you use it to force $P(k+1)$. It is not circular.
- Sum of the first $n$ integers: $n(n+1)/2$. Geometric sum: $(r^{n+1} - 1)/(r - 1)$.
- The same method proves divisibility ($d \mid n$ means $n = d k$, so exhibit the multiple) and inequalities (start the basis where the claim first holds).
- For a recursive sequence, the recurrence supplies the link the inductive step needs to prove a closed form.
- Despite its name, mathematical induction is a deductive method - it proves with certainty, for every n.