Chapter 5 — Sequences & Mathematical Induction

Part 1: 5.1 Sequences

COMP 233 Discrete Mathematics | Birzeit University

Content follows Epp, Discrete Mathematics with Applications (4th ed.).

5.1 Sequences - in this lecture

Why sequences? Patterns are everywhere

A first example: counting ancestors

You have 2 parents, 4 grandparents, 8 great-grandparents, …

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ \ldots$$

$$\text{position } k \ \rightarrow\ \text{value } 2^k$$

Each number is double the one before — a pattern we can name with a formula.

The same idea shows up as:

- Patterns in nature (petals, spirals, branching)
- IQ-test “find the next figure” puzzles
- Train / bus schedules and timetables
- Loops in programs — every loop walks a sequence

5.1 Sequences - in this lecture

What is a sequence?

Definition

A sequence is a set of elements written in a row. Each element is a term.

$$a_m,\ a_{m+1},\ a_{m+2},\ \ldots,\ a_n$$

Reading the notation

a_k is the k-th term. k is the subscript or index. m and n are the lower and upper limits.

Finding terms from an explicit formula

Compute the first terms of each sequence:

Sequence a

$$a_k = \frac{k}{k+1} \quad (k \geq 1)$$

First five terms:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$$

Sequence b

$$b_i = \frac{i-1}{i} \quad (i \geq 2)$$

First four terms:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$$

Alternating sequences

Example

Compute the first six terms of:

$$c_j = (-1)^j \quad (j \geq 0)$$

Solution

$$1,\ -1,\ 1,\ -1,\ 1,\ -1$$

The factor (−1) raised to the index flips the sign at every step — the hallmark of an alternating sequence.

Finding a formula to fit given terms

Find an explicit formula for the sequence

$$1, \; -\frac{1}{4}, \frac{1}{9}, \; -\frac{1}{16}, \frac{1}{25}, \; \ldots$$

Look for two patterns: the size, and the sign

Sizes are 1, 1/4, 1/9, 1/16, … = $1/k^2$. Signs alternate, starting +.

$$a_k = \frac{(-1)^{k+1}}{k^2} \quad (k \geq 1)$$

5.1 Sequences - in this lecture

Summation notation

Definition (the Greek capital sigma Σ means "add")

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

Three things to read off the symbol

- lower limit $k = m$ (where to start)
- upper limit n (where to stop)
- the index k steps up by 1 each term

Computing summations

Example — given specific terms

$$a_1 = -2,\ a_2 = -1,\ a_3 = 0,\ a_4 = 1,\ a_5 = 2$$

$$\sum_{k=1}^{5} a_k = (-2) + (-1) + 0 + 1 + 2 = 0$$

$$\sum_{k=1}^{2} a_{2k} = a_2 + a_4 = -1 + 1 = 0$$

Example — given by a formula

$$\sum_{k=1}^{5} k^2 = 1 + 4 + 9 + 16 + 25 = 55$$

Useful operations on summations

Expanded form	write every term out, ending with “…”
Back to Σ	spot the pattern and fold a long sum into one Σ
Separate / add a term	peel off the last term, or absorb one in
Telescoping	consecutive terms cancel, collapsing the sum

These four moves are exactly how you reshape and optimise loops.

Expanded form ↔ summation

Σ → expanded form

$$\sum_{i=0}^{n}\frac{(-1)^i}{i+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^n}{n+1}$$

expanded form → Σ

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=0}^{n}\frac{k+1}{n+k}$$

Separating off and adding on a final term

Separate off the last term

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

Absorb a loose term into the sum

$$\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

Telescoping sums

Step 1 - split each term into a difference of two pieces

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Step 2 - consecutive pieces cancel, leaving only the two ends

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

first piece − last piece

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

5.1 Sequences - in this lecture

Product notation

Definition (the Greek capital pi Π means "multiply")

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n$$

Example

$$\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

Recursive definition

$$\prod_{k=m}^{m} a_k = a_m, \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n$$

Factorial notation

Definition — n! is the product of all integers from 1 to n

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1, \qquad 0! = 1$$

Recursive definition

$$n! = 1 \quad \text{if } n = 0$$

$$n! = n \cdot (n-1)! \quad \text{if } n \geq 1$$

Why 0! = 1 ?

The recursive rule needs a base case, and 0! = 1 keeps formulas (like counting and the binomial theorem) working cleanly.

Computing with factorials

The trick is always to cancel: write the larger factorial in terms of the smaller.

Numeric

$$\frac{8!}{7!} = \frac{8 \cdot 7!}{7!} = 8$$

$$\frac{5!}{2!\ 3!} = \frac{5 \cdot 4 \cdot 3!}{2!\ 3!} = \frac{20}{2} = 10$$

With a variable n

$$\frac{(n+1)!}{n!} = \frac{(n+1)\, n!}{n!} = n+1$$

$$\frac{n!}{(n-3)!} = n(n-1)(n-2)$$

5.1 Sequences - in this lecture

1. Why we need sequences
2. Sequences and patterns
3. Summation: notation, expanding, telescoping
4. Product and factorial
5. **Properties of summations and products**
6. Sequences in computer loops & change of variable

Properties of summations and products

Theorem 5.1.1 (for sequences and any constant c)

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c\, a_k$$

$$\Big(\prod_{k=m}^{n} a_k\Big)\Big(\prod_{k=m}^{n} b_k\Big) = \prod_{k=m}^{n} (a_k\, b_k)$$

split / combine *pull out a constant* *multiply termwise*

Example — combine into a single summation

$$\sum_{k=m}^{n} a_k + 2\sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (3k - 1)$$

5.1 Sequences - in this lecture

1. Why we need sequences
2. Sequences and patterns
3. Summation: notation, expanding, telescoping
4. Product and factorial
5. Properties of summations and products
6. **Sequences in computer loops & change of variable**

Change of variable (dummy variable)

The index name carries no meaning — it is a dummy variable

$$\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2 = 1 + 4 + 9 = 14$$

Shifting the index: substitute, then fix the limits

$$\sum_{k=0}^{6} \frac{1}{k+1} \qquad = \sum_{j=1}^{7} \frac{1}{j}$$

Sequences in computer loops

A summation is a loop

Σ from k=1 to n of a_k ≡ `s = 0;  for (k = 1; k ≤ n; k++)  s = s + a[k];`

Why this section matters for programming

- The index = the loop counter; limits m, n = the loop bounds.
- Separating a term / changing variable = adjusting loop bounds (off-by-one).
- Telescoping = a loop whose work cancels to a one-line closed form.
- Recursive definitions (product, factorial) = accumulator loops.

Recommended exercises (Epp, 4th ed.)

Write the first terms (explicit formula)	**5.1: 1–6**
Find a formula to fit given terms	**5.1: 10–16**
Compute summations & products	**5.1: 18–28**
Expand / write in Σ–Π notation	**5.1: 29–32, 43–52**
Separate off the final term	**5.1: 37–39**
Combine into a single summation	**5.1: 40–42, 59–61**
Change of variable; factorials	**5.1: 53–58, 62–70**

Summary

- A sequence is an ordered list of terms; the index k marks each term's position.
- An explicit formula gives the k-th term directly; fitting terms to a formula is a conjecture (prove by induction).
- Σ adds terms, Π multiplies them — read the lower limit, upper limit, and index.
- Reshape sums by expanding, separating/adding a term, and telescoping.
- Factorial n! is the key product; use cancellation to simplify, and 0! = 1.
- Every summation, product, and recursive definition is a loop — the bridge to programming.