Chapter 4.2

Direct proof and counterexample - rational numbers

Quotients of integers, and proving their properties

Discrete Mathematics

Hisham Ihshaish

Based on: Epp, Discrete Mathematics with Applications, 4th ed., Section 4.2

Chapter 4.2

Direct proof and counterexample - rational numbers

In this session:

- Part 1: Rational and irrational numbers
- Part 2: Proving properties of rational numbers

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Definition: rational numbers

Definition

A real number r is rational if, and only if, it can be expressed as a quotient of two integers with a nonzero denominator.

A real number that is not rational is called irrational.

More formally, if r is a real number:

r is rational $\Leftrightarrow$ $\exists$ integers a, b such that $r = a / b$ and $b \neq 0$

The word "rational" contains "ratio" - the Latin word for quotient. A rational number is a ratio of integers.

Examples: rational or irrational?

(a) 10 / 3	✓	Rational. Quotient of integers 10 and 3.
(b) −5 / 39	✓	Rational. −5/39 - quotient of −5 and 39.
(c) 0.281	✓	Rational. 0.281 = 281/1000 - any finite decimal is rational.
(d) 7	✓	Rational. 7 = 7/1.
(e) 0	✓	Rational. 0 = 0/1.
(f) 2 / 0	✗	Not a number. Division by zero is undefined.
(g) Is 2 / 0 irrational?	✗	No. Irrational numbers are real numbers; 2/0 is not a number.
(h) 0.12121212... (repeating)	✓	Rational. We will show 0.121212... = 12 / 99.
(i) (m + n) / mn, m, n $\in \mathbb{Z} \setminus \{0\}$.	✓	Rational. m+n and mn are integers; mn ≠ 0 by zero-product property. '\' means EXCEPT (it is a set difference)

Epp, Example 4.2.1, p. 163

Why 0.121212... is rational

Show that 0.12121212... (the digits 12 repeat forever) is a rational number.

Let $x = 0.12121212...$

Then $100x = 12.12121212...$

Subtract: $100x - x = 12.12121212... - 0.12121212... = 12.$

But $100x - x = 99x$ by basic algebra.

Hence $99x = 12$, and so $x = 12 / 99$.

Therefore 0.121212... = 12 / 99, a ratio of two integers with nonzero denominator. So it is rational. ∎

The same trick works for any repeating decimal. In Section 9.4 we show every rational number has a repeating or terminating decimal expansion.

Famous irrational numbers

None of these can be written as a ratio of two integers. We will prove √2 irrational in Section 4.7.

π	3.14159265358979...	*ratio of a circle's circumference to its diameter*
$\sqrt{2}$	1.41421356237309...	*diagonal of a unit square*
$\sqrt{3}$	1.73205080756887...	
$\sqrt{5}$	2.23606797749979...	
$\sqrt{7}$	2.64575131106459...	
$\varphi = (1+\sqrt{5}) / 2$	1.61803398874989...	*the golden ratio* - النّسبة الذّهبية
e	2.71828182845904...	*Euler's number* - العدد النّيبيري (أساس الدالة الأسية الطبيعية)

Theorem 4.2.1 - every integer is rational

The challenge process: "A" claims every integer is rational. "B" challenges with specific cases.

B: Prove it for $n = 7$.

A: $7 = 7/1$, a quotient of integers. Rational.

B: Try $n = -12$.

A: $-12 = -12/1$, a quotient of integers. Rational.

B: How about $n = 0$?

A: $0 = 0/1$, a quotient of integers. Rational.

A's general procedure: given any $n \in \mathbb{Z}$, write $n = n / 1$. This works for every integer.

Theorem 4.2.1

Every integer is a rational number.

Proof. Let n be any integer. Then $n = n / 1$, where n and 1 are integers and $1 \neq 0$.

So n is a quotient of two integers with nonzero denominator. By definition of rational, n is rational. ∎

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Theorem 4.2.2 - sum of rationals is rational

Theorem. The sum of any two rational numbers is rational.

Setting up the proof

Formal restatement. $\forall$ r, s $\in \mathbb{R}$, if r and s are rational then r + s is rational.

Starting point. Suppose r and s are particular but arbitrarily chosen rational numbers.

To show. r + s is rational - i.e., r + s = p/q for some integers p, q with q ≠ 0.

Strategy

Use definition of rational to write r = a/b and s = c/d. Then add the fractions: a/b + c/d = (ad + bc) / bd. Verify the numerator and denominator are integers and the denominator is nonzero.

Theorem 4.2.2 - the proof

Proof

Suppose r and s are rational numbers. [We must show that r + s is rational.]

By definition of rational, $r = a/b$ and $s = c/d$ for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$.

Then

$$r + s = a/b + c/d = (ad + bc) / bd \qquad \text{by basic algebra.}$$

Let $p = ad + bc$ and $q = bd$.

Then p and q are integers (sums and products of integers are integers).

Also $q = bd \neq 0$ by the zero product property (since $b \neq 0$ and $d \neq 0$).

So $r + s = p/q$ where p and q are integers and $q \neq 0$.

Therefore r + s is rational by definition of rational. ∎

Corollary 4.2.3 - double of a rational is rational

Corollary. The double of a rational number is rational.

A corollary is a statement whose truth follows immediately from a theorem already proved.

Proof

Suppose r is any rational number.

Then $2r = r + r$, a sum of two rational numbers.

By Theorem 4.2.2, the sum of two rationals is rational. Hence 2r is rational. ∎

Notice: this proof does not use the definition of rational at all. It cites Theorem 4.2.2 directly.

Deriving new results from old - even/odd properties

Suppose you have already proved the following properties of integers (some are exercises in 4.1):

1. The sum, product, and difference of any two even integers are even.
2. The sum and difference of any two odd integers are even.
3. The product of any two odd integers is odd.
4. The product of any even integer and any odd integer is even.
5. The sum of any odd integer and any even integer is odd.
6. The difference of any odd integer minus any even integer is odd.
7. The difference of any even integer minus any odd integer is odd.

Going forward you may cite any of these by number, instead of re-proving from the definitions.

Example 4.2.3 - try this at home

Example

Use the seven properties from the previous slide to prove:

If a is any even integer and b is any odd integer, then $(a^2 + b^2 + 1) / 2$ is an integer.

Sketch

a is even, so by property 1, $a^2 = a \cdot a$ is even.

b is odd, so by property 3, $b^2 = b \cdot b$ is odd.

By property 5, $a^2 + b^2$ (even + odd) is odd. And 1 is odd. So $(a^2 + b^2) + 1$ is even by property 2.

Hence $a^2 + b^2 + 1 = 2k$ for some integer k. Dividing by 2: $(a^2 + b^2 + 1) / 2 = k$, an integer.

Try the full write-up at home. Cite each property by number where you use it.

Summary

Definition of rational and irrational.

r is rational ⇔ r = a/b for some integers a, b with b ≠ 0. Irrational means real but not rational. 2/0 is not a number at all.

Repeating decimals are rational.

The trick: name the decimal x, multiply by 10^k where k is the block length, subtract, solve for x.

Three results proved today.

Theorem 4.2.1 (every integer is rational), Theorem 4.2.2 (sum of rationals is rational), Corollary 4.2.3 (double of a rational is rational).

New mathematics from old.

Once a result is proved, cite it instead of re-proving. The seven even/odd properties become a toolbox for further proofs.

Recommended exercises

From Epp, Exercise Set 4.2 (pp. 168-169). Grouped by skill.

Identify rationals from a representation *Convert to a/b form, including repeating decimals.*	Exercises 1, 2, 3, 4
Explain why an expression is rational *Use rules of integer arithmetic + zero product property.*	Exercises 9, 10
Direct proofs about rationals *Every integer is rational; negative of a rational is rational; square of a rational is rational.*	Exercises 11, 13, 14
Fill in the blanks in a proof *Guided proof that the square of a rational is rational.*	Exercises 12
True/false with proof or counterexample *Product, quotient, difference, average of rationals.*	Exercises 15, 16, 17, 18
Use the seven even/odd properties *Apply the derived-results method from Example 4.2.3.*	Exercises 21, 22, 23
Find the mistake in a proof *Read a flawed proof and identify the error. High-value for the midterm.*	Exercises 30

Next session

Chapter 4.3

Divisibility - definition and theorems

Definition: d divides n
Transitivity of divisibility
Divisibility by a prime
The unique factorisation theorem