Chapter 3.3

Statements with multiple quantifiers

Order, translation, and negation

Discrete Mathematics

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Based on: Epp, Discrete Mathematics with Applications, 4th ed., Section 3.3

Chapter 3.3

Statements with multiple quantifiers

In this session:

- Part 1: Order of quantifiers and their meaning
- Part 2: Formalizing and verbalizing multiple quantifiers
- Part 3: Negations of multiply-quantified statements

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Order of quantifiers - why it matters

"There is a person supervising every detail of the production process."

Reading A

One supervisor oversees every detail.

$$\exists p \in \mathbf{Person},\ \forall d \in \mathbf{Detail},\ \mathbf{Supervises(p, d)}$$

One fixed person, covers all details.

يوجد شخص واحد يشرف على كل تفاصيل الإنتاج.

Reading B

Every detail has someone supervising it.

$$\forall d \in \mathbf{Detail},\ \exists p \in \mathbf{Person},\ \mathbf{Supervises(p, d)}$$

For each detail, possibly a different person.

لكل تفصيل شخص يشرف عليه (قد يختلف الأشخاص).

Epp, p. 117

Interpreting "for all - there exists" vs "there exists - for all"

$\forall x \in D, \exists y \in E, P(x,y)$

To show it is true:

Someone picks ANY $x \in D$ they wish. Then you must find a $y \in E$ that works for THAT particular x.

Key feature

The y is allowed to DEPEND on x - different x may need a different y.

التحدّي: تختار أنت العنصر الأول، وأنا أجد المناسب.

$\exists x \in D, \forall y \in E, P(x,y)$

To show it is true:

You pick ONE specific $x \in D$ first. Then it must work no matter what $y \in E$ anyone challenges you with.

Key feature

The x is FIXED - it cannot change after someone gives you a y.

التحدّي: تثبيت عنصر واحد أولاً، ثم ينجح مع أي تحدّ.

Epp, p. 119-120

The "loves" table - eight orderings

Formal	Informal meaning
$\forall x\ \forall y \cdot Loves(x,y)$	Everything loves everything.
$\exists x\ \exists y \cdot Loves(x,y)$	Something loves something.
$\forall x\ \exists y \cdot \mathbf{Loves(x,y)}$	Everything loves something (possibly different).
$\exists y\ \forall x \cdot \mathbf{Loves(x,y)}$	Something is loved by everything (one fixed thing).
$\exists x\ \forall y \cdot \mathbf{Loves(x,y)}$	Something loves everything (one lover, many loved).
$\forall y\ \exists x \cdot \mathbf{Loves(x,y)}$	Everything is loved by something (possibly different).

Same-type adjacent quantifiers commute; mixed quantifiers do not.

Movies - six distinct statements

Everyone loves all movies. كل شخص يحبّ كل الأفلام	$\forall p \in \text{Person}, \forall m \in \text{Movie} \cdot \text{Loves}(p, m)$
Some people love some movies. بعض الناس يحبّون بعض الأفلام	$\exists p \in \text{Person}\ \exists m \in \text{Movie} \cdot \text{Loves}(p, m)$
There is a movie that everyone loves. فيلم يحبّه كل الناس	$\exists m \in \text{Movie}\ \forall p \in \text{Person} \cdot \text{Loves}(p, m)$
Some people love all movies. بعض الناس يحبّون كل الأفلام	$\exists p \in \text{Person}\ \forall m \in \text{Movie} \cdot \text{Loves}(p, m)$
Everyone loves some movies. كل شخص يحبّ بعض الأفلام	$\forall p \in \text{Person}\ \exists m \in \text{Movie} \cdot \text{Loves}(p, m)$
All movies are loved by someone. كل فيلم يحبّه بعض الناس	$\forall m \in \text{Movie}\ \exists p \in \text{Person} \cdot \text{Loves}(p, m)$

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Cafeteria example - the setup

A college cafeteria has four stations. Three students go through the line.

Salads	Main courses	Desserts	Beverages
• green salad • fruit salad	• spaghetti • fish	• pie • cake	• milk • soda • coffee

Student choices

Uta: green salad, spaghetti, pie, milk
Tim: fruit salad, fish, pie, cake, milk, coffee
Yuen: spaghetti, fish, pie, soda

Epp, Example 3.3.3, p. 120

Cafeteria - symbolic statements and truth values

(a) $\exists I \in \text{Item}, \forall S \in \text{Student}, \text{Chose}(S, I)$

There is an item that every student chose.

(b) $\exists S \in \text{Student}, \forall I \in \text{Item}, \text{Chose}(S, I)$

There is a student who chose every available item.

(c) $\exists S \in \text{Student}, \forall Z \in \text{Station}, \exists I \in Z, \text{Chose}(S, I)$

There is a student who chose at least one item from every station.

(d) $\forall S \in \text{Student}, \forall Z \in \text{Station}, \exists I \in Z, \text{Chose}(S, I)$

Every student chose at least one item from every station.

Epp, Example 3.3.3, p. 120

Cafeteria - symbolic statements and truth values

(a) $\exists I \in \text{Item}, \forall S \in \text{Student}, \text{Chose}(S, I)$ **TRUE**

There is an item that every student chose.

Every student chose pie.

(b) $\exists S \in \text{Student}, \forall I \in \text{Item}, \text{Chose}(S, I)$ **FALSE**

There is a student who chose every available item.

No student chose all nine items.

(c) $\exists S \in \text{Student}, \forall Z \in \text{Station}, \exists I \in Z, \text{Chose}(S, I)$ **TRUE**

There is a student who chose at least one item from every station.

Uta and Tim both chose from every station.

(d) $\forall S \in \text{Student}, \forall Z \in \text{Station}, \exists I \in Z, \text{Chose}(S, I)$ **FALSE**

Every student chose at least one item from every station.

Yuen did not choose any salad.

Epp, Example 3.3.3, p. 120

What to pay attention to

1. Read outer quantifier first. Every outer quantifier is a challenge or commitment to the whole statement inside.

2. Watch domain restrictions. $\exists I \in Z$ (item from station Z) is not the same as $\exists I \in \text{Item}$. In (c), I must come from station Z, which itself ranges over all stations.

3. Inner ∃ depends on outer ∀. In (c),(d), the item I may change with the student S and station Z. In (a), one item I must work for every student.

4. Do not confuse "∃ then ∀" with "∀ then ∃". (a) says one single item is chosen by every student (∃ first). (d)'s inner ∃I lets each student-station pair choose possibly a different item (∀ first).

Reciprocal of a real number

Definition

The reciprocal (المقلوب الضربي) of a real number a is a real number b such that ab = 1.

(a) Every nonzero real number has a reciprocal.

$$\forall u \in \mathbb{R}_{\neq 0},\ \exists v \in \mathbb{R},\ uv = 1$$

"For each nonzero u, I can find a v (depending on u) with uv = 1." (v = 1/u)

(b) There is a real number with no reciprocal. (The number 0 has no reciprocal.)

$$\exists c \in \mathbb{R},\ \forall d \in \mathbb{R},\ cd \neq 1$$

"There is a real c (namely 0) such that no $d \in \mathbb{R}$ satisfies cd = 1."

Epp, Example 3.3.4, p. 121

Smallest positive integer vs smallest positive real

There IS a smallest positive integer.

$$\exists \mathbf{m} \in \mathbb{Z}^+,\ \forall \mathbf{n} \in \mathbb{Z}^+,\ \mathbf{m} \leq \mathbf{n}$$

Witness: m = 1. For every positive integer n, we have 1 ≤ n.

يوجد أصغر عدد صحيح موجب (وهو 1).

There is NO smallest positive real number.

$$\forall \mathbf{x} \in \mathbb{R}^+,\ \exists \mathbf{y} \in \mathbb{R}^+,\ \mathbf{y} < \mathbf{x}$$

Witness: given any x > 0, take y = x/2. Then 0 < y < x.

لا يوجد أصغر عدد حقيقي موجب - أيّ عدد موجب نجد أصغر منه.

Epp, Examples 3.3.5, 3.3.6, p. 121-122

Tarski's world - formalizing statements

using Formal FOL Notation

Domain D = all objects in the figure. Predicates: Circle, Square, Triangle, Black, Gray, Blue, SameColor, Above, RightOf.

(a) For all circles x, x is above f.

$\forall x \in D,\ Circle(x) \rightarrow Above(x, f)$

(b) There is a square x such that x is black.

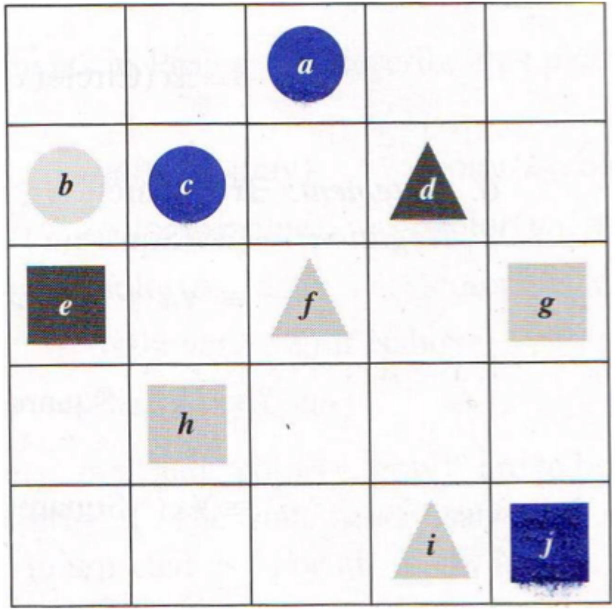
$\exists x \in D,\ Square(x) \wedge Black(x)$

(c) For all circles x, there is a square y such that x and y have the same colour.

$\forall x \in D,\ Circle(x) \rightarrow \exists y \in D, (Square(y) \wedge SameColour(x, y))$

(d) There is a square x such that for all triangles y, x is to the right of y.

$\exists x \in D, Square(x) \wedge (\forall y \in D, Triangle(y) \rightarrow RightOf(x, y))$



Epp, Example 3.3.10, p. 126

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Negation rule - push the tilde stage by stage

From Section 3.2 we have two base rules. Apply them in sequence to multi-quantifier statements.

$$\sim(\forall x \in D, P(x)) \equiv \exists x \in D, \sim P(x)$$

$$\sim(\exists x \in D, P(x)) \equiv \forall x \in D, \sim P(x)$$

Applied to multi-quantifier statements:

$$\sim(\forall x \in D, \exists y \in E, P(x,y)) \equiv \exists x \in D, \forall y \in E, \sim P(x,y)$$

$$\sim(\exists x \in D, \forall y \in E, P(x,y)) \equiv \forall x \in D, \exists y \in E, \sim P(x,y)$$

Epp, p. 123

Negation example - "all people love someone"

Statement. All people love someone.

$\forall x \in \text{Person}, \exists y \in \text{Person}, \text{Loves}(x, y)$

Step 1. Push tilde past ∀:

$$\sim(\forall x, \exists y, \text{Loves}(x, y)) \equiv \exists x, \sim(\exists y, \text{Loves}(x, y))$$

Step 2. Push tilde past ∃:

$$\equiv \exists x, \forall y, \sim\text{Loves}(x, y)$$

Final negation

$\mathbf{\exists x \in Person, \forall y \in Person, \sim Loves(x, y)}$

"There is a person who loves no one." يوجد شخص لا يحبّ أي أحد.

Multiple quantifiers with negated predicates

$\exists x\ \exists y \cdot \sim Loves(x, y)$ يوجد شخص لا يحبّ شخصاً ما.	Somebody does not love somebody.
$\forall x\ \forall y \cdot \sim Loves(x, y)$ لا أحد يحبّ أحداً.	No one loves anyone.
$\exists x\ \forall y \cdot \sim Loves(x, y)$ يوجد شخص لا يحبّ أيّ أحد.	Someone loves no one.
$\forall x\ \exists y \cdot \sim Loves(y, x)$ كل شخص يوجد من لا يحبّه.	Everyone is not loved by someone (everyone has some people who do not love them).

Negating in a Tarski world - worked example

Statement (a)

For all squares x, there is a circle y such that x and y have the same colour.

$\forall x\,(\text{Square}(x) \rightarrow \exists y\,(\text{Circle}(y) \wedge \text{SameColor}(x, y)))$

Negation

There is a square x such that for all circles y, x and y do not have the same colour.

Statement (b)

There is a triangle x such that for all squares y, x is to the right of y.

$\exists x\,(\text{Triangle}(x) \wedge \forall y\,(\text{Square}(y) \rightarrow \text{RightOf}(x, y)))$

Negation

For all triangles x, there is a square y such that x is not to the right of y.

Epp, Example 3.3.8, p. 124

Hold on, how do we negate the first in FOL:

Statement (a)

For all squares x, there is a circle y such that x and y have the same colour.

$\forall x\, (Square(x) \to \exists y\, (Circle(y) \wedge SameColor(x, y)))$

Negation

There is a square x such that for all circles y, x and y do not have the same colour.

Step 1. Apply ¬ to the whole statement and push past ∀x:

$\neg\forall x\, (Square(x) \to \exists y\, (Circle(y) \wedge SameColor(x, y))) \equiv \exists x\, \neg(Square(x) \to \exists y\, (Circle(y) \wedge SameColor(x, y)))$

Step 2. Apply the negation-of-implication rule, $\neg(P \to Q) \equiv P \wedge \neg Q$:

$\equiv \exists x\, (Square(x) \wedge \neg\exists y\, (Circle(y) \wedge SameColor(x, y)))$

Step 3. Push ¬ past ∃y, flipping it to ∀y:

$\equiv \exists x\, (Square(x) \wedge \forall y\, \neg(Circle(y) \wedge SameColor(x, y)))$

Epp, Example 3.3.8, p. 124

Negation - more English examples

Everyone loves someone.	**There is someone who loves no one.**
$\forall x\ \exists y \cdot \text{Loves}(x, y)$	$\exists x\ \forall y \cdot \sim\text{Loves}(x, y)$
Some person loves everyone.	**Everyone fails to love at least one person.**
$\exists x\ \forall y \cdot \text{Loves}(x, y)$	$\forall x\ \exists y \cdot \sim\text{Loves}(x, y)$
Every positive real has a reciprocal.	**Some positive real has no reciprocal.**
$\forall u \in \mathbb{R}^+, \exists v \in \mathbb{R} \cdot uv = 1$	$\exists u \in \mathbb{R}^+, \forall v \in \mathbb{R} \cdot uv \neq 1$
There is a program that solves every instance.	**For every program, some instance is not solved by it.**
$\exists p\ \forall i \cdot \text{Solves}(p, i)$	$\forall p\ \exists i \cdot \sim\text{Solves}(p, i)$

Summary

Order of quantifiers matters.

Swapping ∀∃ with ∃∀ usually changes the meaning. ∃∀ is stronger than ∀∃.

∀ before ∃ → inner can depend on outer.

In ∀x ∃y, the y is allowed to be a function of x. In ∃x ∀y, x is fixed before y is seen.

Restriction pattern.

"∀ cats c, P(c)" translates to ∀c (Cat(c) → P(c)). "∃ cat c such that P(c)" translates to ∃c (Cat(c) ∧ P(c)).

Negation rule.

To negate a multiply-quantified statement: flip every quantifier (∀↔∃) and negate the innermost predicate.