Chapter 3.2

Predicates and quantified statements II

Negation and conditional statements

Discrete Mathematics

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Based on: Epp, Discrete Mathematics with Applications, 4th ed., Section 3.2:

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Chapter 3.2

Predicates and quantified statements II

In this session:

- Part 1: Negations of quantified statements
- Part 2: Contrapositive, converse, and inverse
- Part 3: Necessary and sufficient conditions, only if

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The common mistake

How would you negate this?

"All mathematicians wear glasses."

Wrong

"No mathematicians wear glasses."

This says every single mathematician is a non-wearer - a much stronger claim than needed.

Correct

"There is at least one mathematician who does not wear glasses."

One counterexample is enough to make "all" false.

Negation of a universal statement

Theorem 3.2.1

The negation of a statement of the form

$$\forall x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D \text{ such that } {\sim}Q(x)$$

Symbolically:

$$\sim(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } {\sim}Q(x)$$

"all are" becomes "some are not"

Negation of an existential statement

Theorem 3.2.2

The negation of a statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \sim Q(x)$$

Symbolically:

$$\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x)$$

"some are" becomes "none are" ("all are not")

Epp, Theorem 3.2.2, p. 109

Negation - worked examples

Original statement	Correct negation
All Palestinians like Zaatar. ∀p ∈ Palestinian, Likes(p, Zaatar)	Some Palestinians do not like Zaatar. ∃p ∈ Palestinian such that ~Likes(p, Zaatar)
Some Palestinians like Zaatar. ∃p ∈ Palestinian such that Likes(p, Zaatar)	No Palestinian likes Zaatar. ∀p ∈ Palestinian, ~Likes(p, Zaatar)
Every prime is odd. ∀p ∈ Prime, Odd(p)	There is a prime that is not odd. ∃p ∈ Prime such that ~Odd(p)
Some computer hackers are over 40. ∃h ∈ Hacker such that Over40(h)	No computer hacker is over 40. ∀h ∈ Hacker, ~Over40(h)
All computer programs are finite. ∀c ∈ Program, Finite(c)	Some computer program is not finite. ∃c ∈ Program such that ~Finite(c)

Epp, Example 3.2.1, p. 110

Negating a universal conditional

Most statements in mathematics have the form $\forall x, P(x) \rightarrow Q(x)$. How do we negate them?

Step 1. Apply Theorem 3.2.1:

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x))$$

Step 2. From Chapter 2, $\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x)$:

$$\exists x \text{ such that } P(x) \wedge \sim Q(x)$$

Negation of a universal conditional statement

$$\sim(\forall x, \text{if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x)$$

In words: "there is at least one x that satisfies the hypothesis but fails the conclusion."

Negating conditionals - examples

Original statement	Negation
If a computer program has more than 10,000 lines, then it contains a bug. $\forall c$, ManyLines(c) → HasBug(c)	There is a computer program with more than 10,000 lines that does not contain a bug. $\exists c$ such that ManyLines(c) $\wedge$ ~HasBug(c)
If a person is blond, then they have blue eyes. $\forall p \in$ Person, Blond(p) → BlueEyes(p)	There is a blond person who does not have blue eyes. $\exists p \in$ Person such that Blond(p) $\wedge$ ~BlueEyes(p)
No politicians are honest. $\forall p \in$ Politician, ~Honest(p)	Some politician is honest. $\exists p \in$ Politician such that Honest(p)

Caution. The negation of "if P then Q" does ***not*** start with "if". It starts with "there is".

Epp, Example 3.2.4, p. 111

Vacuous truth

Definition

A statement of the form $\forall$**x** $\in$ **D, if P(x) then Q(x)** is called ***vacuously true*** (or true by default) if, and only if, P(x) is false for every x in D.

Why?

The negation is $\exists x \in D$ such that $P(x) \wedge \sim Q(x)$. If no x makes P(x) true, the negation cannot be witnessed, so the original is true "by default".

Example

Place no balls in an empty bowl. The statement "all the balls in the bowl are blue" is vacuously true - there is no non-blue ball to contradict it.

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Contrapositive, converse, inverse

Given the universal conditional $\forall x \in D, P(x) \rightarrow Q(x)$

Contrapositive	Converse	Inverse
$\forall x \in D,$ $\sim Q(x) \rightarrow \sim P(x)$	$\forall x \in D,$ $Q(x) \rightarrow P(x)$	$\forall x \in D,$ $\sim P(x) \rightarrow \sim Q(x)$
Flip and negate both.	*Just flip.*	*Just negate both.*
≡ original	**≢ original**	**≢ original**
always logically equivalent	*not equivalent in general*	*not equivalent in general*

Epp, Definition, p. 113

Example 1 - Palestinian and smart

Original

$\forall x \in \mathbf{Person},\ \mathbf{Palestinian}(x) \rightarrow \mathbf{Smart}(x)$

Contrapositive	$\forall x \in \text{Person},\ \sim\text{Smart}(x) \rightarrow \sim\text{Palestinian}(x)$
Converse	$\forall x \in \text{Person},\ \text{Smart}(x) \rightarrow \text{Palestinian}(x)$
Inverse	$\forall x \in \text{Person},\ \sim\text{Palestinian}(x) \rightarrow \sim\text{Smart}(x)$

Read aloud: original says "every Palestinian is smart." The converse says "every smart person is Palestinian" - a very different claim.

Example 2 - squares greater than 4

Original

$\forall x \in \mathbb{R},\ \ \mathbf{x > 2 \ \rightarrow\ x^2 > 4}$

Contrapositive	$\forall x \in \mathbb{R},\ \ x^2 \le 4 \ \rightarrow\ x \le 2$
Converse	$\forall x \in \mathbb{R},\ \ x^2 > 4 \ \rightarrow\ x > 2$
Inverse	$\forall x \in \mathbb{R},\ \ x \le 2 \ \rightarrow\ x^2 \le 4$

Counterexample to the converse: $x = -3$ gives $x^2 = 9 > 4$ but $x = -3 \ngtr 2$. So the converse is false while the original is true.

Epp, Example 3.2.5, p. 113

Why the contrapositive is equivalent

Let P(x), Q(x) be predicates over domain D. Compare:

Original

$$\forall x \in D,\ P(x) \to Q(x)$$

Contrapositive

$$\forall x \in D,\ {\sim}Q(x) \to {\sim}P(x)$$

From Chapter 2: $p \to q \equiv {\sim}q \to {\sim}p$

So for each fixed x in D, "P(x) → Q(x)" and "~Q(x) → ~P(x)" have the same truth value.

Therefore both universal statements hold for the same x values, so they are logically equivalent.

$$\forall x \in D,\ P(x) \to Q(x) \quad \equiv \quad \forall x \in D,\ {\sim}Q(x) \to {\sim}P(x)$$

Epp, p. 114

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Necessary, sufficient, ~~only if~~

Definition

"**∀x, r(x) is a sufficient condition for s(x)**" means "∀x, r(x) → s(x)"

"**∀x, r(x) is a necessary condition for s(x)**" means "∀x, ~r(x) → ~s(x)"

equivalently: "∀x, s(x) → r(x)"

"**∀x, r(x) only if s(x)**" means "∀x, ~s(x) → ~r(x)" ≡ "∀x, r(x) → s(x)"

Intuition

Sufficient: having r is enough to guarantee s. r → s.

Necessary: s cannot happen without r. s → r.

~~Only if: r happens only if s is in place first. r → s.~~

Epp, Definition, p. 114

Sufficient conditions - examples

Being a square is a sufficient condition for being a rectangle.

In other words: if a figure is a square, then it is a rectangle.

$$\forall x, \ \ \text{Square}(x) \ \rightarrow \ \text{Rectangle}(x)$$

Squareness guarantees rectangularity - it's more than enough.

To get a job, it is sufficient to be loyal.

In other words: if a person is loyal, then they will get the job.

$$\forall p, \ \ \text{Loyal}(p) \ \rightarrow \ \text{GetsJob}(p)$$

Loyalty alone suffices - other people might still get the job for other reasons.

Epp, Example 3.2.6(a), p. 115

Necessary conditions - examples

Being smart is a necessary condition for getting a job.

Informal: if you are not smart, you do not get a job - equivalently - if you got a job, you must be smart.

$$\forall p,\ \sim Smart(p) \rightarrow \sim GetsJob(p) \quad \equiv \quad \forall p,\ GetsJob(p) \rightarrow Smart(p)$$

Being smart does not guarantee the job - but you can't get the job without it.

Being at least 35 years old is necessary for being President of the United States.

Informal: if you are under 35, you cannot be President - or - if you are President, you are at least 35.

$$\forall p,\ Under35(p) \rightarrow \sim President(p) \quad \equiv \quad \forall p,\ President(p) \rightarrow \geq 35 \text{ years old}$$

Age 35+ is required, but not sufficient - millions of 35+ year-olds are not President.

Epp, Example 3.2.6(b), p. 115

~~"Only if" in practice~~

You get the job only if you are at the top.

Meaning: getting the job happens only when being at the top is already in place.

$$\forall p,\ \sim\text{Top}(p) \rightarrow \sim\text{GetsJob}(p) \quad \equiv \quad \forall p,\ \text{GetsJob}(p) \rightarrow \text{Top}(p)$$

Informal: if you are not at the top, you will not get the job. Equivalently, if you got the job, you were at the top.

Example 3.2.7 from Epp

"A product of two numbers is 0 only if one of the numbers is 0."

Rewritten as a universal conditional:

$$\forall a, b \in \mathbb{R},\ ab = 0 \rightarrow a = 0 \text{ or } b = 0$$

Epp, Example 3.2.7, p. 115