

Chapter 3.1

# Predicates and quantified statements

*First order logic - introduction*

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Discrete Mathematics

Hisham Ihshaish

*Based on: Epp, Discrete Mathematics with Applications, 4th ed., Section 3.1*

# Predicates and quantified statements

In this lecture:

**Part 1: What is a predicate, and predicate logic**

Part 2: Universal and existential quantifiers:  $\forall, \exists$

Part 3: Formalize and verbalize statements

Part 4: Tarski's World

# What is first order logic?

Propositional Logic	First Order Logic
P, Q Propositions	P(x), Q(x,y) Predicates
$\neg P$ Negation $P \wedge Q$ Conjunction $P \vee Q$ Disjunction $P \rightarrow Q$ Implication $P \leftrightarrow Q$ Equivalence	$\neg P$ Negation $P \wedge Q$ Conjunction $P \vee Q$ Disjunction $P \rightarrow Q$ Implication $P \leftrightarrow Q$ Equivalence $\forall$ Universal quantification $\exists$ Existential quantification
<b>We regard the world as Propositions</b>	<b>We regard the world as Quantified Predicates</b>

**Also called:** Predicate Logic, First-Order Predicate Calculus, Quantification Theory

# What is a predicate?

A **predicate** is a sentence that contains a finite number of variables and becomes a **statement** when specific values are substituted for the variables. The **domain** is the set of all values that may be substituted.

$$P(x_1, x_2, \dots, x_n)$$

## Examples

**Person(Amjad)**

Amjad is a person

*unary*

**University(BZU)**

BZU is a university

*unary*

**StudyAt(Amjad, BZU)**

Amjad studies at BZU

*binary*

# Truth set of a predicate

If  $P(x)$  is a predicate and  $x$  has domain  $D$ , the **truth set** of  $P(x)$  is the set of all elements of  $D$  that make  $P(x)$  true:  $\{x \in D \mid P(x)\}$

## Examples

$\{x \in \text{Organization} \mid \text{University}(x)\}$

The set of all organizations that are universities.

$\{x \in \text{Person} \mid \text{Student}(x)\}$

The set of all persons that are students.

Let  $Q(n) = 'n \text{ is a factor of } 8'$ , domain =  $Z^+$ . Truth set =  $\{1, 2, 4, 8\}$ .

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# Universal statements ( $\forall$ )

A **universal statement** has the form ' $\forall x \in D, Q(x)$ '. It is true if  $Q(x)$  is true for **every**  $x$  in  $D$ . It is false if  $Q(x)$  is false for **at least one**  $x$  in  $D$  (a **counterexample**).

## Examples

$$\forall x \in \mathbf{R}, x^2 \geq x$$

False - counterexample:  $x = 0.5$ , then  $0.25 < 0.5$

Let  $D = \{1, 2, 3, 4, 5\}$

$$\forall x \in \mathbf{D}, x^2 \geq x$$

True - check:  $1 \geq 1, 4 \geq 2, 9 \geq 3, 16 \geq 4, 25 \geq 5$

$\triangle$  The domain matters! The same predicate  $x^2 \geq x$  is false over  $\mathbf{R}$  but true over  $\{1,2,3,4,5\}$ .

# Existential statements ( $\exists$ )

An **existential statement** has the form ' $\exists x \in D, Q(x)$ '. It is true if  $Q(x)$  is true for **at least one**  $x$  in  $D$ . It is false if  $Q(x)$  is false for **all**  $x$  in  $D$ .

## Examples

$$\exists m \in \mathbf{Z}^+, m^2 = m$$

True -  $m = 1$  works:  $1^2 = 1$

Let  $E = \{5, 6, 7, 8\}$

$$\exists m \in E, m^2 = m$$

False - none of 5,6,7,8 satisfies  $m^2 = m$

# $\forall$ and $\exists$ as $\wedge$ and $\vee$

$$\forall x \in D, Q(x) \Leftrightarrow Q(x_1) \wedge Q(x_2) \wedge \dots \wedge Q(x_n)$$

$$\exists x \in D, Q(x) \Leftrightarrow Q(x_1) \vee Q(x_2) \vee \dots \vee Q(x_n)$$

$\forall$  is a **big AND**: every element must satisfy  $Q(x)$ . If any one fails, the whole conjunction is false.

$\exists$  is a **big OR**: at least one element must satisfy  $Q(x)$ . If any one succeeds, the whole disjunction is true.

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# Verbalizing formal statements

Write the following formal statements in informal language:

$$\forall x \in \mathbf{R}, x^2 \geq 0$$

The square of every real number is non-negative.

$$\forall x \in \mathbf{R}, x^2 \neq -1$$

The square of any real number does not equal -1.

$$\exists m \in \mathbf{Z}^+, m^2 = m$$

There is a positive integer that equals its square.

$$\forall x \in \mathbf{R}, x > 2 \rightarrow x^2 > 4$$

For every real number, if it is greater than 2 then its square is greater than 4.

# Formalizing statements

Write the following informal statements in formal language:

**All triangles have three sides.**

$\forall t \in \text{Triangle}, \text{ThreeSided}(t)$

**No dogs have wings.**

$\forall d \in \text{Dog}, \neg \text{HasWings}(d)$

**Some programs are structured.**

$\exists p \in \text{Program}, \text{Structured}(p)$

**All bytes have eight bits.**

$\forall b \in \text{Byte}, \text{EightBits}(b)$

**No fire trucks are green.**

$\forall t \in \text{FireTruck}, \neg \text{Green}(t)$

⚠ 'No X are Y' is formalized as  $\forall x \in X, \neg Y(x)$  - not  $\exists$ .

# Universal conditional statements

The most common form in mathematics:

$$\forall x \in D, P(x) \rightarrow Q(x)$$

## Example

If a real number is an integer, then it is a rational number.

$$\forall n \in \text{RealNumber}, \text{Integer}(n) \rightarrow \text{Rational}(n)$$

## Equivalent forms:

$$\forall x \in \text{Square, Rectangle}(x)$$

$\forall x$ , if  $x$  is a square then  $x$  is a rectangle

$\forall$  squares  $x$ ,  $x$  is a rectangle

# Implicit quantification

Quantifiers are sometimes hidden in everyday language.

**If a number is an integer, then it is rational.**

$\forall n, \text{Integer}(n) \rightarrow \text{Rational}(n)$

*implicit*  $\forall$

**If a person was born in Hebron then s/he is Khalili.**

$\forall x \in \text{Person}, \text{BornIn}(x, \text{Hebron}) \rightarrow \text{Khalili}(x)$

*implicit*  $\forall$

**People who like hummus are smart.**

$\forall x \in \text{Person}, \text{Like}(x, \text{Hummus}) \rightarrow \text{Smart}(x)$

*implicit*  $\forall$

The word 'a' or 'any' often signals implicit universal quantification. 'Some' or 'there is' signals existential.

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# Tarski's World

Tarski's World is a tool for evaluating first order logic statements against a visible grid of objects with shapes and colours.

*Predicates: Triangle(x), Square(x), Blue(x), Gray(x), RightOf(x,y)*

Given the world in Epp Figure 3.1.1, evaluate:

$\forall t, \text{Triangle}(t) \rightarrow \text{Blue}(t)$

✓ True

$\forall x, \text{Blue}(x) \rightarrow \text{Triangle}(x)$

✗ False (e is blue, not triangle)

$\exists y, \text{Square}(y) \wedge \text{RightOf}(d, y)$

✓ True (e, h)

$\exists z, \text{Square}(z) \wedge \text{Gray}(z)$

✗ False (squares are blue/black)

# Summary of key concepts

Concept	Symbol	Meaning	True when...
Predicate	$P(x)$	Statement with variables	values substituted
Truth set	$\{x \in D \mid P(x)\}$	Values making P true	by definition
Universal	$\forall x \in D, Q(x)$	For all x in D, Q(x)	Q(x) true for every x
Existential	$\exists x \in D, Q(x)$	There exists x in D, Q(x)	Q(x) true for at least one x
Univ. conditional	$\forall x, P(x) \rightarrow Q(x)$	If P(x) then Q(x) for all x	no x has P true and Q false

# Key takeaways

## ■ Predicates are statements with variables

$P(x)$  becomes a statement when  $x$  gets a specific value

## ■ Two quantifiers turn predicates into statements

$\forall$  (for all) and  $\exists$  (there exists)

## ■ $\forall$ is a big AND, $\exists$ is a big OR

One counterexample kills  $\forall$ ; one witness proves  $\exists$

## ■ 'No X are Y' uses $\forall$ with $\neg$ , not $\exists$

Common formalization mistake to watch for

## ■ The domain determines truth

The same predicate can be true or false depending on the domain

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**Next session:** Chapter 3.2 - Negation and conditional statements with quantifiers