Chapter 2.3

Valid and invalid arguments

Inference rules and logical reasoning

Discrete Mathematics

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Based on: Epp, Discrete Mathematics with Applications, 4th ed., Section 2.3

Chapter 2.3

Valid and invalid arguments

In this lecture:

Part 1: Arguments and the numeration method

Part 2: Rules of inference

Part 3: Worked example

Part 4: Fallacies and soundness

What is an argument?

An **argument** is a sequence of statements called **premises** followed by a statement called the **conclusion**.

The symbol $\therefore$ means "therefore" and introduces the conclusion.

Example

If today is Friday, then today is a holiday.	$p \to q$	*premise (major)*
Today is Friday.	p	*premise (minor)*
$\therefore$ **Today is a holiday.**	q	*conclusion*

An argument consisting of two premises and a conclusion is called a **syllogism**. The first premise is the **major premise** and the second is the **minor premise**.

Valid vs invalid arguments

An argument form is **valid** if whenever all premises are true, the conclusion is also true.

An argument form is **invalid** if there exists a case where all premises are true but the conclusion is false.

Testing validity with a truth table

1. List all premises and the conclusion as column headers.
2. Fill in the truth table for all combinations of variable values.
3. Find every **critical row** - a row where ALL premises are true.
4. If the conclusion is true in every critical row → valid.
5. If the conclusion is false in any critical row → invalid.

Numeration method - example 1

Argument: **p → q, p, ∴ q**

premises *conclusion*

p	q	p → q	p	q
T	T	T	T	T
T	F	F	T	
F	T	T	F	
F	F	T	F	

← **critical row**

Only one critical row (row 1): both premises are true and the conclusion is true.
The argument form is **valid**. This form is called **modus ponens**.

Numeration method - example 2

Argument: **p → q ∨ ~r, q → p ∧ r, ∴ p → r**

p	q	r	~r	q∨~r	p∧r	p→q∨~r	q→p∧r	p→r
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

← critical row: conclusion is F

Row 4: all premises true but conclusion false. The argument is **invalid**.

Why we need inference rules

The truth table method always works, but it does not scale.

Variables (n)	Rows (2^n)
2	4
3	8
5	32
10	1,024
20	1,048,576

The solution: rules of inference

Pre-verified argument forms that we can apply directly, without building a truth table each time.

Each rule has been proven valid once and for all. We use them as building blocks to construct complex chains of reasoning.

A **rule of inference** is a form of argument that has been proven valid. We can apply it whenever we recognise the pattern.

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Modus ponens

Rule 1

$p \to q$
p
$\therefore$ **q**

Example

Let p = 'today is Friday', q = 'today is a holiday'

If today is Friday then today is a holiday.

Today is Friday.

$\therefore$ **Today is a holiday.**

Modus tollens

Rule 2

$p \rightarrow q$
$\sim q$
$\therefore \sim p$

Example

Let p = 'today is Friday', q = 'today is a holiday'

If today is Friday then today is a holiday.

Today is not a holiday.

∴ **Today is not Friday.**

Generalization

Rule 3

p

$\therefore$ **$p \vee q$**

Example

Let p = 'today is Saturday', q = 'today is Sunday'

Today is Saturday.

$\therefore$ **Today is Saturday or today is Sunday.**

Specialization

Rule 4

$p \wedge q$
$\therefore$ **p**

Example

Let p = 'today is Friday', q = 'today is a holiday'

Today is Friday and today is a holiday.

$\therefore$ **Today is Friday.**

Conjunction

Rule 5

p
q
$\therefore\ p \wedge q$

Example

Let p = 'today is Friday', q = 'today is a holiday'

Today is Friday.

Today is a holiday.

∴ **Today is Friday and today is a holiday.**

Elimination

Rule 6

$p \vee q$
$\sim p$
$\therefore$ **q**

Example

Let p = 'today is Saturday', q = 'today is Sunday'

Today is Saturday or today is Sunday.

Today is not Saturday.

∴ **Today is Sunday.**

Transitivity

Rule 7

$p \rightarrow q$
$q \rightarrow r$
$\therefore p \rightarrow r$

Example

Let p = 'today is Friday', q = 'today is a holiday', r = 'I am happy'

If today is Friday then today is a holiday.

If today is a holiday then I am happy.

∴ **If today is Friday then I am happy.**

Division into cases

Rule 8

$p \vee q$
$p \to r$
$q \to r$
$\therefore$ **r**

Example

Let p = 'today is Friday', q = 'today is Sunday', r = 'I am happy'

Today is Friday or today is Sunday.

If today is Friday then I am happy.

If today is Sunday then I am happy.

$\therefore$ **I am happy.**

Contradiction rule

Rule 9

$\sim p \rightarrow c$
$\therefore$ **p**

Example

Let p = 'today is Friday', c = contradiction

If 'today is not Friday' leads to a contradiction (a statement always false).

$\therefore$ **Today is Friday.**

Summary of inference rules

Rule	Premises	Conclusion	Rule	Premises	Conclusion
Modus Ponens	$p \to q, p$	$\therefore$ **q**	**Elimination**	$p \vee q, \sim p$	$\therefore$ **q**
Modus Tollens	$p \to q, \sim q$	$\therefore$ **~p**	**Transitivity**	$p \to q, q \to r$	$\therefore$ **p→r**
Generalization	p	$\therefore$ **p∨q**	**Div. into Cases**	$p \vee q, p \to r, q \to r$	$\therefore$ **r**
Specialization	$p \wedge q$	$\therefore$ **p**	**Contradiction**	$\sim p \to c$	$\therefore$ **p**
Conjunction	p, q	$\therefore$ **p∧q**			

Memorise these nine rules. They are the building blocks for all deductive proofs in Chapters 3 and 4.

Epp, Section 2.3, Table 2.3.1

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Inferencing example - problem

Where are the glasses?

(a) If I was reading in the kitchen, then my glasses are on the kitchen table.	**$RK \to GK$**
(b) If my glasses are on the kitchen table, then I saw them at breakfast.	**$GK \to SB$**
(c) I did not see my glasses at breakfast.	**$\sim SB$**
(d) I was reading in the living room or in the kitchen.	**$RL \lor RK$**
(e) If I was reading in the living room, then my glasses are on the coffee table.	**$RL \to GC$**

Variables: RK = reading in kitchen, GK = glasses on kitchen table, SB = saw at breakfast, RL = reading in living room, GC = glasses on coffee table

Inferencing example - solution

Variables:
RK = reading in kitchen
GK = glasses on kitchen table
SB = saw at breakfast
RL = reading in living room
GC = glasses on coffee table

Step 1 *Transitivity*

(a) RK → GK, (b) GK → SB

∴ **RK → SB**

Step 2 *Modus Tollens*

RK → SB (step 1), (c) ~SB

∴ **~RK**

Step 3 *Elimination*

(d) RL ∨ RK, ~RK (step 2)

∴ **RL**

Step 4 *Modus Ponens*

(e) RL → GC, RL (step 3)

∴ **GC**

Conclusion: the glasses are on the coffee table.

Epp, Section 2.3, Example 2.3.8

Fallacies and soundness

Common errors that resemble valid reasoning

Converse error

$p \rightarrow q,\ q,\ \therefore p$ ← INVALID

If it rains, the ground is wet.
The ground is wet.
∴ It rained. ✗ (sprinkler?)

Affirming the consequent

Inverse error

$p \rightarrow q,\ {\sim}p,\ \therefore {\sim}q$ ← INVALID

If you study, you pass.
You did not study.
∴ You did not pass. ✗ (easy exam?)

Denying the antecedent

Summary

- **An argument is valid if no critical row has a false conclusion**

  Validity is about form, not the truth of individual statements

- **Nine inference rules replace truth tables**

  Modus ponens, modus tollens, generalization, specialization, conjunction, elimination, transitivity, division into cases, contradiction

- **Fallacies mimic valid patterns**

  Converse error and inverse error look like modus ponens and tollens but are invalid

- **Sound = valid + all premises true**

  Only sound arguments guarantee the conclusion is true

- **Inference rules chain together**

  Complex arguments are built step by step from simple rules

Next session: Chapter 3.1 - Predicates and quantified statements