Discrete Mathematics

Section 2.2

Conditional statements

Based on: Discrete Mathematics with Applications, 4th Ed. - Susanna S. Epp, Section 2.2

Keywords: conditional, implication, hypothesis, conclusion, contrapositive, converse, inverse, biconditional, necessary, sufficient, only if

If-then statements

Definition

If p and q are statement variables, the **conditional** of q by p is "If p then q" or "p implies q", denoted $\mathbf{p \rightarrow q}$.

It is false when p is true and q is false; otherwise it is true.

We call p the **hypothesis** (or antecedent) and q the **conclusion** (or consequent).

- Epp, Section 2.2, p. 39

Example:

If 4,686 is divisible by 6, then 4,686 is divisible by 3

hypothesis — conclusion

If you study, then you pass

Truth table for $p \to q$

The job interview promise

A store owner tells you: "If you show up Monday morning, then you get the job."

When can you say the owner lied?

p	q	$p \to q$	Scenario
T	T	T	You showed up, got the job
T	F	F	You showed up, didn't get it
F	T	T	Didn't show up, got it anyway
F	F	T	Didn't show up, didn't get it

← **The owner lied!**

The only way $p \to q$ is **false** is when p is **true** and q is **false**.

A false hypothesis cannot break a promise.

Conditional with a false hypothesis

If 0 = 1, then 1 = 2.

This statement is true.

A conditional statement with a false hypothesis is called **vacuously true**.

We do not test the correctness of the conclusion - **we only check whether the hypothesis being true forces the conclusion to be true.**

More examples:

- "If dogs can fly, then 2 + 2 = 5" is **true**
- "If the moon is made of cheese, then I am the president" is **true**

Truth table exercise: p ∨ ~q → ~p

Construct the truth table for the statement form p ∨ ~q → ~p

p	q	~p	~q	p ∨ ~q	p ∨ ~q → ~p
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Observation: This statement form is **not a tautology** - it is false in rows 1 and 2.

Remember: → has **lower precedence** than ∨ and ∧, so p ∨ ~q is the hypothesis and ~p is the conclusion.

Logical equivalences involving →

Division into cases:

$p \lor q \to r \equiv (p \to r) \land (q \to r)$

p	q	r	p∨q	p→r	q→r	p∨q→r	(p→r)∧(q→r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

The last two columns are identical → **logically equivalent.**

Representation of if-then as or

$$p \rightarrow q \equiv \sim p \vee q$$

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example:

"If it rains, the match is cancelled"

is equivalent to:

"It does not rain, or the match is cancelled"

Negation of a conditional statement

Theorem

$\sim(p \rightarrow q) \equiv p \wedge \sim q$

The negation of "if p then q" is logically equivalent to "p and not q."

Derivation:

$\sim(p \rightarrow q)$

$\equiv \sim(\sim p \vee q)$ by if-then as or

$\equiv \sim(\sim p) \wedge (\sim q)$ by De Morgan's law

$\equiv p \wedge \sim q$ by double negation law

Caution: the negation does NOT start with "if"!

Negation of conditional - examples

Example 1:

Statement: "If my lecture is at Masri109, then I cannot buy coffee"

Negation: **"My lecture is at Masri109 and I can buy coffee"**

Example 2:

Statement: "If Sara studies hard, then she passes the exam"

Negation: **"Sara studies hard and she does not pass the exam"**

Pattern: $\sim(p \rightarrow q) \equiv p \wedge \sim q$

The hypothesis stays the same, the conclusion is negated, and "if...then" becomes "and".

The contrapositive

Definition

The **contrapositive** of a conditional statement "**If p then q**" is:

"If ~q then ~p" i.e., $\sim q \rightarrow \sim p$

A conditional statement is logically equivalent to its contrapositive.

- Epp, Section 2.2, p. 43

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

Example:

Statement: "If you study, then you pass"

Contrapositive: **"If you do not pass, then you did not study"**

Contrapositive - examples

Example 1:

Statement: "If Ali can swim across the lake, then Ali can swim to the island"

Contrapositive: **"If Ali cannot swim to the island, then Ali cannot swim across the lake"**

Example 2:

Statement: "If today is Easter, then tomorrow is Monday"

Contrapositive: **"If tomorrow is not Monday, then today is not Easter"**

Verify with truth table:

$p \to q$ and $\sim q \to \sim p$ produce identical truth values in all four rows. **They are logically equivalent.**

Epp, Section 2.2, Example 2.2.6, p. 43

Converse and inverse

Definition

Given a conditional statement "If p then q":

1. The **converse** is "If q then p" i.e., $q \to p$
2. The **inverse** is "If ~p then ~q" i.e., $\sim p \to \sim q$

- Epp, Section 2.2, p. 44

Summary of all four forms:

Form	Symbolic	Arabic
Original (conditional)	$p \to q$	عبارة شرطية
Contrapositive	$\sim q \to \sim p$	عكس النقيض
Converse	$q \to p$	المقلوب
Inverse	$\sim p \to \sim q$	المعكوس

Converse and inverse - examples

Statement: "If Ali can swim across the lake, then Ali can swim to the island"

Converse: "If Ali can swim to the island, then Ali can swim across the lake"

Inverse: "If Ali cannot swim across the lake, then Ali cannot swim to the island"

Statement: "If today is Easter, then tomorrow is Monday"

Converse: "If tomorrow is Monday, then today is Easter"

Inverse: "If today is not Easter, then tomorrow is not Monday"

Note: The original statement about Easter is always true, but both the converse and inverse are **false** on every Sunday except Easter.

Epp, Section 2.2, Example 2.2.7, p. 44

Caution - common misconception

Many people believe that if a conditional statement is true, then its converse and inverse must also be true.
This is not correct!

- Epp, Section 2.2, p. 44

1. A conditional statement and its **converse** are **NOT** logically equivalent.

2. A conditional statement and its **inverse** are **NOT** logically equivalent.

3. The **converse** and the **inverse** of a conditional statement **ARE** logically equivalent to each other.

The inverse is the contrapositive of the converse, hence they are equivalent.

"If" vs. "only if"

Not included: ignore if you want

Let p = "you get a diploma" and q = "you pass all exams"

"p if q"

"if" marks q as the hypothesis:

$q \to p$

"You get a diploma **if** you pass all exams."

Pass → diploma. Passing is **sufficient.**

"p only if q"

p cannot happen without q:

$p \to q$

"You get a diploma **only if** you pass all exams."

Diploma → passed. Passing is **necessary.**

The arrows point in opposite directions!

Why does "p only if q" give $p \to q$? Read it as: without q, no p. That is $\sim q \to \sim p$.

By contrapositive: $\sim q \to \sim p \equiv p \to q$. Same statement, different intuition.

Combine both: "p if and only if q" = $(q \to p) \wedge (p \to q) = p \leftrightarrow q$

Epp, Section 2.2, p. 45

The biconditional (↔)

Definition

Given statement variables p and q, the **biconditional** of p and q is "p if, and only if, q", denoted **$p \leftrightarrow q$**. It is true if both p and q have the same truth values, and false if p and q have opposite truth values.

The words "if and only if" are sometimes abbreviated ***iff***.

- Epp, Section 2.2, p. 46

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Think of it as: both statements agree.

Biconditional as two conditionals

$$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$$

p	q	p→q	q→p	p↔q	(p→q)∧(q→p)
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

The last two columns are identical → **the biconditional equals the conjunction of both directions.**

Example: "This program is correct **iff** it produces correct answers for all inputs" means:

(1) If correct → produces correct answers   AND   (2) If produces correct answers → correct

Necessary and sufficient conditions

Definition

If r and s are statements:

- "r is a **sufficient condition** for s" means "if r then s" $(r \to s)$
- "r is a **necessary condition** for s" means "if not r then not s" $(\sim r \to \sim s)$
  which is equivalent to "if s then r" $(s \to r)$

- "r is a **necessary and sufficient condition** for s" means $r \leftrightarrow s$

- Epp, Section 2.2, p. 47-48

Hint:

Sufficient = enough to guarantee it happens

Necessary = cannot happen without it

Necessary and sufficient - examples

Example 1: Studying and passing

"Studying is a sufficient condition for passing" → **Study → Pass**

"In order to pass, it is sufficient to study" → **Study → Pass**

"Studying is a necessary condition for passing" → **~Study → ~Pass**

Example 2: Age and ID card

"Being above 16 is a sufficient condition for getting an ID card"

Above(16) → Get ID Card

"Being above 16 is a necessary condition for getting an ID card"

~Above(16) → ~Get ID Card

Order of operations and summary

Order of operations for logical operators:

Priority	Operator	Rule
1 (highest)	~	Evaluate negations first
2	∧, ∨	Evaluate AND and OR second
3 (lowest)	→, ↔	Evaluate conditional and biconditional last

- Epp, Section 2.2, p. 48

Summary from Section 2.2:

- p → q is false only when p is true and q is false
- p → q ≡ ~p ∨ q (if-then as or)
- ~(p → q) ≡ p ∧ ~q (negation of conditional)
- p → q ≡ ~q → ~p (contrapositive - equivalent)
- Converse (q → p) and inverse (~p → ~q) are NOT equivalent to the original
- p ↔ q ≡ (p → q) ∧ (q → p) (biconditional)
- Sufficient = r → s; Necessary = ~r → ~s (equivalently s → r)